Risk Matters: Breaking Certainty Equivalence in Linear Approximations*

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September 2020

Abstract

In this paper we use the property that certainty equivalence, as implied by a first-order approximation to the solution of stochastic discrete-time models, breaks in its equivalent continuous-time version. We study the extent to which a first-order approximated solution built by perturbation methods accounts for risk. We show that risk matters economically in a real business cycle (RBC) model with habit formation, and capital adjustment costs and that neglecting risk leads to substantial pricing errors. A first-order approximation in continuous time reduces pricing errors by 90 percent relative to the certainty equivalent linear solution.

Keywords: Certainty equivalence, Perturbation methods, Pricing errors
JEL classification: C02, C61, C63, E13, E32, G12.

*Declarations of interest: None. We thank Serguei Mailiar, Johannes Pfeiffer, Alexander Meyer-Gohde, Martin M. Andreasen, participants to the 23rd International Conference in Computing in Economics and Finance (New York, 2017), the 13th Dynare Conference (Tokyo, 2017), the 11th International Conference on Computational and Financial Econometrics (London, 2017), the 26th Annual SNDE Symposium (Tokyo, 2018), the 23rd Annual LACEA Meeting (Guayaquil, 2018), the 6th International Conference on Economics of the Turkish Economic Association (Antalya, 2018); and seminar participants for their valuable comments and discussions. We appreciate financial support from the German Research Foundation (DFG, Project No. 268475812). This version supersedes an earlier working paper entitled "Risk Matters: Breaking Certainty Equivalence".

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1 Introduction

There is a consensus among economists that uncertainty affects the consumption-saving decision of individuals. Neglecting the effects of risk in macroeconomics and finance often generates substantial pricing errors. Hence, recent research is concerned with the ability of local approximations of non-linear stochastic macroeconomic models to account for risk, with a particular focus on perturbation methods originally introduced in economics by Judd and Guu (1993). Although perturbation-based methods only provide local precision around a particular point, usually the model's deterministic steady state, many authors suggest that they can generate high levels of accuracy, comparable to that delivered by global approximation techniques, as the order of the approximation is increased (see Judd, 1998; Aruoba et al., 2006; Caldara et al., 2012; Parra-Alvarez, 2018). In many applications, however, we are interested in the first-order perturbation and the resulting linear approximation of the equilibrium conditions. A linear structure not only provides analytical insights and helps to understand key features of the model, but also facilitates its estimation, e.g., by means of the Kalman filter (see Harvey and Stock, 1985; Singer, 1998; Harvey, 2006; Fernández-Villaverde and Rubio-Ramírez, 2007), and ensures non-explosive forecasts and simulations without the need to impose explicit conditions on the support of the shocks driving the economy.

A known limitation of the first-order perturbation around the deterministic steady state is that the approximate solution of discrete time models typically exhibits certainty equivalence (see Simon, 1956; Theil, 1957). In other words, the first-order approximation to the solution of stochastic economic models with forward-looking agents is identical to the solution of the same model under perfect foresight. The direct implication is that the solution becomes invariant to higher-order moments of the underlying shocks. Therefore, this paper addresses the following questions. What are the costs of neglecting the effects of risk in linear approximations? Put differently, what would be the benefits of using a linear approximation that is not certainty equivalent? In particular, by how much could such an approximation reduce the errors that one makes when not accounting for risk? How can these errors be interpreted in an economically meaningful way?

Certainty equivalence prevails in the classical linear-quadratic optimal control problem, popularized in economics by Kydland and Prescott (1982) and Anderson et al. (1996). In the early 1950s the introduction of certainty equivalent stochastic control problems with quadratic utility and linear constraints aimed at providing a practical solution for decision problems under uncertainty. Even today, if risk is negligible for the research question at hand, certainty equivalent solutions are still useful. In this case, one may conclude that "certainty equivalence is a virtue" (see Kimball, 1990a). Conversely, when there is a reason to believe that the effects of risk are important, one notices that "certainty equivalence is a vice". Put differently, if risk matters, breaking certainty equivalence is desired in order to account for the effects of risk. As discussed in Fernández-Villaverde et al. (2016), the approximated solution of the model under certainty equivalence (i) makes it difficult to talk about the welfare effects of uncertainty; (ii) cannot generate any risk premia for assets; and (iii) prevents analyzing the consequences of changes in volatility.

To break the property of certainty equivalence in the class of perturbation methods, economists have restored to the computation of higher-order Taylor expansions, the underlying apparatus behind any perturbation method, which translate into non-linear approximations of the model's solution. Originally proposed in Judd and Guu (1993), higher-order approximations became popular with the work of Schmitt-Grohe and Uribe (2004) for second-order approximations, and that of Andreasen (2012) and Ruge-Murcia (2012) for third-order approximations. More recently, Levintal (2017) extended the perturbation package to include fifth-order approximations. However, the use of high-order approximations for medium-scale macroeconomic models (i) could be computationally expensive, (ii) often results in explosive solutions, and (iii) requires computationally demanding non-linear estimation methods, such as the particle filter, for the estimation of the model's structural parameters.

In contrast to stochastic discrete-time models, certainty equivalence breaks in the first-order approximation when time is assumed to be continuous (see Judd, 1996; Gaspar and Judd, 1997). This property, which allows us to account for risk in a linear world, is the product of two complementary results. First, while in discrete time the approximation is built inside the system of expectational equations that collects the equilibrium conditions of the economy, in continuous time we may use Itô's lemma to eliminate the expectation operator prior to the construction of the approximation. The resulting *non-expectational* system of equations, although deterministic, will capture the effects of uncertainty by including information on the sensibility to risk of the yet unknown solution (see Chang, 2009). Second, as shown in Fleming (1971), who provides the mathematical foundations of perturbation methods for continuous-time stochastic optimal control problems, regular perturbation theory produces asymptotically valid approximations of the unknown solution when the variance of the shocks is used as perturbation parameter. As discussed in Jin and Judd (2002), this choice of the perturbation parameter is not arbitrary but instead follows basic economic intuition, whereby *"the economic response should be proportional to the variance"*. This is in contrast to discrete-time models where the appropriate perturbation parameter is shown to be the standard deviation (cf. Judd, 1998, Jin and Judd, 2002, Fernández-Villaverde et al., 2016). Combining these two results, the linear approximation to the model's solution, which results from a first-order perturbation around the deterministic steady state, will exhibit a constant correction term that depends on the variance of the shocks that drive the dynamics of the economy.

In this paper we revisit the ability of a first-order perturbation to capture the effects of risk. Using as a benchmark an RBC model with habit formation and capital adjustment costs *à la* Jermann (1998), we compare how the effects of uncertainty are internalized by perturbations built around the model's deterministic steady state relative to a more accurate solution obtained by projection methods. First, we build approximated solutions to the continuous-time model and show that the certainty equivalence property already breaks in the first order such that the associated linear approximation is risk-sensitive. More specifically, the first-order perturbation will correct for risk through an additional constant term that incorporates information on the slope and curvature of the optimal policy functions at the deterministic steady state. We then calibrate the parameters of the model to values that are standard in the literature and compare, along different dimensions, the first-order certainty equivalent (CE) solution to the first- and second-order approximations obtained from perturbation. We show that each of the approximations converges to different long-run equilibria or fixed points in the absence of shocks. While the first-order CE converges to the deterministic steady state, risk-adjusted solutions converge to their respective risky steady states. This property is reflected in the policy and impulse response functions.

We find that the risk effects captured by the first-order approximation in continuous time are economically significant. We assess the asset pricing implications of the approximations using a partial differential equation approach rather than the standard simulation approach used in discrete time. When relying on the linear CE solution the pricing errors are about 1 dollar for each 100 dollar spent. The risk-adjustment of the first-order perturbation approximation leads to errors of about 10 cents for each 100 dollar spent, reducing pricing errors by about 90%. In the second-order perturbation approximation, pricing errors fall further to about 3 cents. We also find that the continuous-time first-order approximation is especially useful in situations in which risk matters but nonlinearities are negligible.

Our work relates to that of Collard and Juillard (2001), Coeurdacier et al. (2011), de Groot (2013), Meyer-Gohde (2015) and Lopez et al. (2018), who compute first-order approximations around the model's risky steady state instead of the deterministic steady state in order to break certainty equivalence in discrete-time models. Collard and Juillard (2001) consider a bias reduction procedure to compute the approximation around the risky steady state; Coeurdacier et al. (2011), whose approach is generalized by de Groot (2013), approximate the risky steady state based on the second-order solution. Meyer-Gohde (2015) constructs a risk-sensitive linear approximation using policy functions resulting from higher-order perturbations. Lopez et al. (2018) differ from the previous studies as they consider lognormal affine approximations, often used in macro-finance, which are shown to be a special case of a first-order perturbation around the risky steady state. We argue that it is possible to account for risk in an economically meaningful way using *standard* first-order (*linear*) perturbations around the *deterministic* steady state when time evolves continuously.

The rest of the paper is organized as follows. In Section 2, we introduce our model and define the equilibrium conditions used in the perturbation method to approximate the solution. Section 3 summarizes the perturbation approach and revisits the property of certainty equivalence in linear models. Section 4 derives the pricing implications of the approximated solution and introduces a pricing error measure that can be used to evaluate the accuracy of the approximation. Section 5 discusses the main results by comparing policy functions, impulse-response functions, and pricing errors for different degrees of approximation. Finally, Section 6 concludes.

2 A prototype RBC model

For illustration, we use a continuous-time version of the real business cycle model (RBC) introduced in Jermann (1998) with some minor modifications. There is a single good in the economy that is produced using a constant-returns-to-scale production technology that is subject to random shocks in productivity. Changes in the economy's aggregate capital stock are subject to adjustment costs, and the household preferences exhibit intertemporal non-separabilities due to internal habit formation in consumption. A discrete-time version of the model can be found in accompanying web appendix.

Preferences. The economy is inhabited by a large number of identical households that maximize their expected discounted lifetime utility from consumption, C_t,

$$U_0 \equiv \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u\left(C_t, X_t\right) \mathrm{d}t \right], \tag{1}$$

where $\mathbb{E}_0[\cdot]$ is the expectation operator conditional on the information available at time $t = 0$, $\rho \geq 0$ is the household's subjective discount rate, and u is the instantaneous utility function. For simplicity we assume that

$$u\left(C, X\right) = \frac{\left(C - X\right)^{1-\gamma}}{1-\gamma}, \tag{2}$$

where γ measures the curvature of the utility function (together with the consumption surplus ratio) so, *ceteris paribus*, a higher value of γ yields higher risk aversion. In what follows, we assume that the consumption choice is non-negative, $C_t \geq 0$, and does not fall below a subsistence level of consumption, $C_t \geq X_t$, where X_t denotes habits in consumption. Hence, the instantaneous utility in (1) is said to exhibit adjacent complementarity in consumption (see Ryder and Heal, 1973), where an increase in consumption today increases the marginal utility of consumption at adjacent dates relative to the marginal utility of consumption at distant ones. The habit level in consumption is defined endogenously (internal habit) in the model, in contrast to the relative consumption model or 'catching up with the Joneses' (external habit), where the habit is aggregate consumption and thus exogenous to the households. In particular, the habit process is given by

$$X_t = e^{-at}X_0 + b\int_0^t e^{a(s-t)}C_s \mathrm{d}s, \quad X_0 \geq 0,$$

or equivalently,

$$\mathrm{d}X_t = (bC_t - aX_t)\mathrm{d}t. \tag{3}$$

Hence, X_t is a weighted sum of past consumption, with weights declining exponentially into the past. The larger is b, the less weight is given to past consumption in determining X_t and vice versa. The special case $b = X_0 = 0$ corresponds to the case of time-separable utility with constant relative risk aversion (see Constantinides, 1990). The parameter a measures the degree of persistence in the habit stock.

Technology. The one good in the economy is produced according to a Cobb-Douglas production function

$$Y_t = \exp(A_t)K_t^{\alpha}L_t^{1-\alpha}, \quad 0 < \alpha < 1, \tag{4}$$

where K_t is the aggregate capital stock, L_t is the perfectly inelastic labor supply (normalized to one for all $t \geq 0$), and A_t is a stochastic process representing random total factor productivity (TFP). The aggregate capital stock in the economy increases if effective investment exceeds depreciation

$$\mathrm{d}K_t = (\Phi(I_t/K_t) - \delta)K_t\mathrm{d}t, \quad K_0 > 0, \tag{5}$$

where $\delta \geq 0$ is the depreciation rate, and I_t is aggregate investment. Following Jermann (1998), the capital adjustment cost function is defined by

$$\Phi(I_t/K_t) = \frac{a_1}{1-1/\xi}\left(I_t/K_t\right)^{1-1/\xi} + a_2, \tag{6}$$

where $\xi > 0$ denotes the elasticity of the investment-to-capital ratio with respect to Tobin's q, and $a_1 \geq 0$ and $a_2 \geq 0$ are parameters. In line with Boldrin et al. (2001), we set $a_1 = \delta^{1/\xi}$ and $a_2 = \delta/(1-\xi)$ such that the steady state is invariant to ξ, and hence the long-run investment-to-capital ratio equals the deprecation rate[1]. On the other hand, TFP is assumed to follow an Ornstein-Uhlenbeck process with mean reversion $\rho_A > 0$ and variance $\sigma_A > 0$

$$\mathrm{d}A_t = -\rho_A A_t\mathrm{d}t + \sigma_A\mathrm{d}B_{A,t}, \tag{7}$$

where $B_{A,t}$ is a standard Brownian motion. In equilibrium, the economy satisfies the

[1]Given this parameterization it can be shown that in the steady state: $\Phi(\bar{I}/\bar{K}) = \Phi(\delta) = \delta$, $\Phi'(\bar{I}/\bar{K}) = \Phi'(\delta) = 1$, and $\Phi''(\bar{I}/\bar{K}) = \Phi''(\delta) = -1/(\xi\delta)$, i.e. the slope of Φ' depends negatively on ξ and δ.

aggregate resource constraint

$$Y_t = C_t + I_t. \tag{8}$$

Optimality conditions. Consider the problem faced by a social planner who has to choose the path for consumption that maximizes (1) subject to the dynamic constraints (3), (5), and (7), and the static constraints (4), (6), and (8)

$$V(K_0, X_0, A_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} U_0 \quad \text{s.t.} \quad (3) - (8), \tag{9}$$

in which C_t is the control variable at time $t \in \mathbb{R}^+$, and $V(K_0, X_0, A_0) \equiv V_0$ is the value of the optimal plan (value function) from the perspective of time $t = 0$, i.e., when the state of the economy is described by the time $t = 0$ values for the capital stock, K_0, the stock of habits, X_0, and the total factor productivity, A_0.

As shown in Appendix A, for any $t \in [0, \infty)$, a necessary condition for optimality is given by the *Hamilton-Jacobi-Bellman* (HJB) equation

$$0 = \max_{C \geq X \in \mathbb{R}^+} \left\{ \frac{(C - X)^{1-\gamma}}{1 - \gamma} + (\Phi((\exp(A)K^\alpha - C)/K)K - \delta K)V_K \right. \\ \left. + (bC - aX)V_X - \rho_A A V_A + \tfrac{1}{2}\sigma_A^2 V_{AA} - \rho V \right\}, \tag{10}$$

where $V_K \equiv \partial V(K, X, A)/\partial K$, $V_X \equiv \partial V(K, X, A)/\partial X$ and $V_A \equiv V(K, X, A)/\partial A$, and $V_{AA} \equiv \partial^2 V(K, X, A)/\partial A^2$ denote, respectively, the first-order partial derivatives and the second-order partial derivative of the value function with respect to the states of the economy[2]. The first order condition for any interior solution reads

$$(C - X)^{-\gamma} + bV_X = \Phi' \left(\frac{\exp(A)K^\alpha - C}{K} \right) V_K, \tag{11}$$

making optimal consumption an implicit function of the state variables, $C^\star = C(K, X, A)$. The function $C(\cdot)$ maps every possible values of the states of the economy at time t into the optimal consumption at time t. The maximized HJB equation reads

$$0 = \frac{(C(K, X, A) - X)^{1-\gamma}}{1 - \gamma} + (\Phi((\exp(A)K^\alpha - C(K, X, A))/K)K - \delta K)V_K \\ + (bC(K, X, A) - aX)V_X - \rho_A A V_A + \tfrac{1}{2}\sigma_A^2 V_{AA} - \rho V, \tag{12}$$

which together with the first order condition (11) determine the unknown functions $V(K, X, A)$ and $C(K, X, A)$ that define the equilibrium in the economy.

[2] A formal introduction and derivation of the dynamic programming equation for continuous-time problems can be found in Chang (2009). In what follows, we omit the use of the time index given the recursive structure of the HJB equation.

Equilibrium dynamics. A solution to the continuum of problems formed by (11) and (12) can be characterized in the *time-space domain* by a sequence $\{V_{K,t}, V_{X,t}, K_t, X_t, A_t\}_{t=0}^{\infty}$ that solves the boundary value problem (with appropriate transversality conditions) characterized by the system of equilibrium stochastic differential equations (SDEs)

$$\begin{aligned}
\mathrm{d}V_{K,t} &= (\rho - \Phi((\exp(A_t)K_t^{\alpha} - C_t)/K_t) - \Phi'((\exp(A_t)K_t^{\alpha} - C_t)/K_t) \\
&\quad \times[(\alpha-1)\exp(A_t)K_t^{\alpha-1} + C_t/K_t] + \delta)V_{K,t}\mathrm{d}t + V_{KA,t}\sigma_A \mathrm{d}B_{A,t} && (13)\\
\mathrm{d}V_{X,t} &= ((\rho+a)V_{X,t} + (C_t - X_t)^{-\gamma})\mathrm{d}t + V_{XA,t}\sigma_A\mathrm{d}B_{A,t} && (14)\\
\mathrm{d}K_t &= \left(\Phi\left((\exp(A_t)K_t^{\alpha} - C_t\right)/K_t\right)K_t - \delta K_t\right)\mathrm{d}t && (15)\\
\mathrm{d}X_t &= (bC_t - aX_t)\mathrm{d}t && (16)\\
\mathrm{d}A_t &= -\rho_A A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}, && (17)
\end{aligned}$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$ and where C_t solves the non-linear algebraic equation in (11).

Alternatively, we may eliminate time and shocks from the system of equilibrium SDEs in (13)-(17) and define the solution to the optimal control problem in the *state-space domain* as the triple $\{V_K(K,X,A), V_X(K,X,A), C(K,X,A)\}$ for admissible values of the state space (K, X, A) that solves the system of partial differential equations (PDEs)

$$\begin{aligned}
0 &= (\rho - \Phi((\exp(A)K^{\alpha} - C)/K) - \Phi'((\exp(A)K^{\alpha} - C)/K)((\alpha-1)\exp(A)K^{\alpha-1} \\
&\quad + C/K) + \delta)V_K - (\Phi((\exp(A)K^{\alpha} - C)/K)K - \delta K)V_{KK} \\
&\quad - (bC - aX)V_{XK} + \rho_A A V_{AK} - \tfrac{1}{2}\sigma_A^2 V_{AAK} && (18)\\
0 &= (\rho + a)V_X + (C - X)^{-\gamma} - (\Phi((\exp(A)K^{\alpha} - C)/K)K - \delta K)V_{KX} \\
&\quad - (bC - aX)V_{XX} + \rho_A A V_{AX} - \tfrac{1}{2}\sigma_A^2 V_{AAX} && (19)\\
0 &= (C - X)^{-\gamma} + bV_X - \Phi'((\exp(A)K^{\alpha} - C)/K)V_K. && (20)
\end{aligned}$$

A complete derivation of both the equilibrium system of SDEs in the time-space domain, and its PDE representation in the state-space domain can be found in Appendix A.

Deterministic steady state. In the absence of uncertainty (i.e. $\sigma_A = 0$), the economy converges over time to a fixed point or steady-state equilibrium in which all variables are idle. Given the assumptions on the capital adjustment cost function in (6), the deterministic steady state of the model is fully characterized by

$$\overline{A} = 0 \tag{21}$$
$$\overline{K} = [\alpha/(\rho+\delta)]^{\frac{1}{1-\alpha}} \tag{22}$$
$$\overline{C} = \overline{K}^{\alpha} - \delta\overline{K} \tag{23}$$
$$\overline{X} = (b/a)\overline{C} \tag{24}$$
$$\overline{V}_X = -[1/(\rho+a)]\left(\overline{C}-\overline{X}\right)^{-\gamma} \tag{25}$$
$$\overline{V}_K = [1-b/(\rho+a)]\left(\overline{C}-\overline{X}\right)^{-\gamma}, \tag{26}$$

where $\overline{V}_X$ and $\overline{V}_K$ are the deterministic steady state values of the costate variables for the capital stock and the habit formation. For a detailed derivation of the model's deterministic steady state see Appendix A.

3 Approximate solution

Most dynamic economic models do not admit an analytical solution, so it usually has to be approximated using numerical methods (see Fernández-Villaverde et al., 2006). Perturbation methods are fast and reliable, and provide an approximate solution to the stochastic optimal control problem in (9) based on the implicit function theorem and the Taylor's series expansion theorem. The perturbed solution consists of a polynomial that approximates the true solution of the problem locally in a neighborhood of an *a priori* known solution. In what follows, we build the perturbation solution to the equilibrium system of PDEs in (18)-(20) around the deterministic steady state given by (21)-(26).

Let $\eta > 0$ denote a *perturbation parameter* that rescales the amount of risk in the economy. For continuous-time stochastic optimal control problems, Fleming (1971) showed that by choosing η to control the *variance* of the exogenous disturbances, it is possible to use regular perturbation theory to obtain asymptotically valid approximations to the unknown policy functions (see Judd, 1996; Gaspar and Judd, 1997)[3]. As shown below, choosing instead η to control *the standard deviation* of the exogenous shocks would result in certainty equivalent approximations, missing the effects of risk. Therefore, the exogenous stochastic processes for TFP (7) is rewritten as

$$\mathrm{d}A_t = -\rho_A A_t \mathrm{d}t + \sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t},$$

where the case $\eta = 0$ makes the model deterministic, and $\eta = 1$ recovers the true TFP

[3] This is in contrast to discrete-time stochastic problems, where the perturbation parameter rescales the *standard deviation* of the shocks (see Schmitt-Grohe and Uribe, 2004; Fernández-Villaverde et al., 2016). Choosing instead the variance of the shocks as the perturbation parameter produces approximations with undesirable stochastic properties as shown in Jin and Judd (2002).

process in (7). Following Judd (1998), the perturbation method can be summarized as follows:

1. Express the problem of interest as a continuum of problems parameterized by the added perturbation parameter η, with the $\eta = 0$ case known.

2. Differentiate the continuum of problems with respect to the state variables and the perturbation parameter η.

3. Solve the resulting equation for the implicitly defined derivatives at the known solution of the state variables and $\eta = 0$.

4. Compute the desired order of approximation by means of Taylor's theorem. Set $\eta = 1$ to recover the approximation to the original model.

In what follows, we introduce a general framework for the perturbation method for continuous-time models[4]. We then provide an illustrative example of the method by using a simplified version of our prototype model. Subsequently, we explain why the property of certainty equivalence that usually results from any first-order perturbation approximation to discrete-time models breaks in continuous-time models. Finally, we introduce the notion of the risky steady state, which will become relevant for understanding transition paths from the model.

3.1 Solving the model: A general framework

In continuous time, the general equilibrium can be represented by the functional equation

$$\mathcal{H}\left(\mathbf{y}, \mathbf{y}_{\mathbf{x}}, \mathbf{y}_{\mathbf{xx}}, \mathbf{x}; \eta\right) = \mathbf{0}, \tag{27}$$

where $\mathbf{y}$ denotes the vector of control (or costate) variables, $\mathbf{y}_{\mathbf{x}}$ and $\mathbf{y}_{\mathbf{xx}}$ the matrices of first- and second-order partial derivatives of the control variables with respect to the vector of state variables $\mathbf{x}$, η is the perturbation parameter, and $\mathcal{H}$ is a functional operator collecting the model's equilibrium conditions. The state vector $\mathbf{x}$ evolves over time according to the controlled stochastic differential equation

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}\left(\mathbf{x}_t, \mathbf{y}_t\right)\mathrm{d}t + \sqrt{\eta}\mathbf{G}\left(\mathbf{x}_t\right)\mathrm{d}\mathbf{B}_t, \tag{28}$$

where $\mathbf{f}(\cdot)$ is the drift vector, $\mathbf{G}(\cdot)$ is a diffusion matrix, potentially dependent on the current value of the state vector, and $\mathbf{B}_t$ is the vector of exogenous shocks.

[4] A description of the perturbation method for discrete-time models can be found in the accompanying web appendix and the references therein.

A solution to the functional equation in (27) takes the form

$$\mathbf{y} = \mathbf{g}\left(\mathbf{x};\eta\right), \tag{29}$$

where $\mathbf{g}\left(\cdot\right)$ is a vector of policy functions that maps every possible value of $\mathbf{x}$ into $\mathbf{y}$.

The deterministic steady state of the model in (27) is defined as the tuple $\left(\overline{\mathbf{y}}, \overline{\mathbf{y}}_{\mathbf{x}}, \overline{\mathbf{y}}_{\mathbf{xx}}, \overline{\mathbf{x}}\right)$ that satisfies

$$\mathcal{H}\left(\overline{\mathbf{y}}, \overline{\mathbf{y}}_{\mathbf{x}}, \overline{\mathbf{y}}_{\mathbf{xx}}, \overline{\mathbf{x}}; 0\right) = \mathbf{0}. \tag{30}$$

Accordingly, it follows that in the steady state $\overline{\mathbf{y}} = \mathbf{g}\left(\overline{\mathbf{x}};0\right)$.

A perturbation-based approximation to the solution of the problem in (27) starts by plugging the unknown solution (29) into the functional equation $\mathcal{H}$ to obtain the new operator

$$F\left(\mathbf{x};\eta\right) := \mathcal{H}\left(\mathbf{g}\left(\mathbf{x};\eta\right), \mathbf{g}_{\mathbf{x}}\left(\mathbf{x};\eta\right), \mathbf{g}_{\mathbf{xx}}\left(\mathbf{x};\eta\right), \mathbf{x};\eta\right) = \mathbf{0}, \tag{31}$$

where $\mathbf{g}_{\mathbf{x}}$ is the matrix of first-order partial derivatives of the policy function, and $\mathbf{g}_{\mathbf{xx}}$ is the matrix of second-order partial derivatives[5]. The perturbation approximation exploits the fact that if $F\left(\mathbf{x};\eta\right) = \mathbf{0}$ for any admissible values of $\mathbf{x}$ and η, then its derivatives must also be zero. That is, $F_{x^k,\eta^j}\left(\mathbf{x};\eta\right) = \mathbf{0}$, for all $x \in \mathbf{x}, \eta, k, j$, where $F_{x^k,\eta^j}\left(\mathbf{x};\eta\right)$ denotes the k-th derivative of F with respect to the state variable $x \in \mathbf{x}$, and with respect to η taken j times, evaluated at $\left(\mathbf{x};\eta\right)$.

Let $k = 1$. Then, the *First-Order* approximation around the deterministic steady state to the policy functions is defined by

$$\mathbf{g}(\mathbf{x};\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}-\overline{\mathbf{x}}) + \mathbf{g}_{\eta}(\overline{\mathbf{x}};0), \tag{32}$$

where the coefficient $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$ corresponds to the stable solution to the quadratic matrix-equation that results from solving $F_{\mathbf{x}}\left(\overline{\mathbf{x}};0\right) = \mathbf{0}$. Similarly, $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0)$ is the solution to the linear system of equations $F_{\eta}\left(\overline{\mathbf{x}};0\right) = \mathbf{0}$. As opposed to perturbation of stochastic discrete-time models, this constant is not necessarily zero in continuous time when the perturbation is built on the variance of the shocks (see Judd, 1996). Therefore, the First-Order approximation in (32) includes a correction term that captures the effects of risk, i.e., it is risk-sensitive, and hence it does *not* exhibit certainty equivalence. By setting $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0) = \mathbf{0}$ in (32) we define the corresponding *First-Order Certainty Equivalent* (CE) approximation.

Now let $k = 2$. Then, the *Second-Order* approximation to the unknown policy function

[5]As opposed to discrete-time models, the solution to the type of continuous-time models considered here does not require the approximation of a policy function for the next period state variables. However, their values at a given point in time can be recovered, *ex-post*, by solving the corresponding (controlled) differential equations in (28).

around the deterministic steady state is defined by

$$\mathbf{g}(\mathbf{x};\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}-\overline{\mathbf{x}}) + \mathbf{g}_{\eta}(\overline{\mathbf{x}};0) \\ + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}-\overline{\mathbf{x}}) \otimes (\mathbf{x}-\overline{\mathbf{x}}) + \mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}-\overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0) \quad (33)$$

where the remaining unknown coefficients are obtained as the solution to the linear system of equations formed by $F_{x_i x_j}(\overline{\mathbf{x}};0) = \mathbf{0}$ for all $x_i, x_j \in \mathbf{x}$, $F_{x_i \eta}(\overline{\mathbf{x}};0) = \mathbf{0}$ for all $x_i \in \mathbf{x}$, and $F_{\eta\eta}(\overline{\mathbf{x}};0) = \mathbf{0}$. In contrast to the discrete-time case, all the coefficients from the Second-Order approximation are different from zero. Hence, it provides an additional source of risk corrections beyond that already introduced through $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0)$. While the latter, together with $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$, only affect the level of the policy functions, $\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)$ adjusts their slopes, introducing in this way a time-varying risk correction component already in a second-order approximation.

As discussed in Judd and Guu (1993) and Judd (1998), the risk-correction term in (32) for noncertainty equivalent economies, requires information on the slope and curvature of the optimal policy functions at the deterministic steady state, i.e. $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0) = \mathbf{g}_{\eta}(\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0), \mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0))$. Note that despite the need for second-order information, the approximated policy function remains linear in the states. Any higher-order information needed in the computation of the risk-correction terms can be immediately and accurately computed from the deterministic version of the model. For the First-Order perturbation, this amounts to compute $\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$ directly from the solution to the linear system of equations formed by $F_{x_i x_j}(\mathbf{x};0) = \mathbf{0}$ for all $x_i, x_j \in \mathbf{x}$ evaluated at $(\overline{\mathbf{x}};0)$[6]. Although the first-order approximation for continuous-time models is more involved than the first-order approximation for discrete-time models, it is less involved than a second-order approximation for discrete-time models.

3.2 An illustration: The stochastic growth model

To illustrate how the procedure described above works, consider the stochastic neoclassical growth model which results from setting $X_0 = b = 0$ and letting $\xi \to \infty$ in the prototype model of Section 2. The HJB equation to the planner's problem is

$$\rho V(K,A;\eta) = \max_{C \in \mathbb{R}^+} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{\mathrm{d}t}\mathbb{E}_t\Big[\mathrm{d}V(K,A;\eta)\Big] \right\}$$

[6]In general, knowledge of the first k derivatives of $\mathbf{g}(\mathbf{x};\eta)$ with respect to $\mathbf{x}$, only provides information to compute the first $(k-2)$ derivatives of $\mathbf{g}_{\eta}(\mathbf{x};\eta)$ with respect to $\mathbf{x}$. For the Second-Order approximation in (33), the computation of $\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)$ and $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$ requires information on $\mathbf{g}_{\mathbf{xxx}}(\overline{\mathbf{x}};0)$ and $\mathbf{g}_{\mathbf{xxxx}}(\overline{\mathbf{x}};0)$, besides that already provided by $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$ and $\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$.

where the aggregate capital stock and TFP evolve according to:

$$\begin{aligned} \mathrm{d}K_t &= \left(\exp(A_t)K_t^{\alpha} - C_t - \delta K_t\right)\mathrm{d}t, \quad K_0 > 0, && (34)\\ \mathrm{d}A_t &= -\rho_A A_t \mathrm{d}t + \sqrt{\eta\sigma_A^2}\,\mathrm{d}B_{A,t}, \quad A_0 > 0. && (35) \end{aligned}$$

Using Itô's lemma together with the properties of stochastic integrals, we can write the HJB equation as

$$\rho V(K, A; \eta) = \max_{C\in\mathbb{R}^+} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + (\exp(A)K^{\alpha} - C - \delta K)V_K(K, A; \eta) - \rho_A A V_A(K, A; \eta) + \tfrac{1}{2}\eta\sigma_A^2 V_{AA}(K, A; \eta) \right\}. \quad (36)$$

A complete derivation can be found in Appendix B. The equilibrium of the economy can be characterized in the time-space domain by the sequence $\{C_t, K_t, A_t\}_{t=0}^{\infty}$ that solves the system of SDEs formed by the Euler equation for consumption

$$\frac{\mathrm{d}C_t}{C_t} = \left[\frac{\alpha\exp(A_t)K_t^{\alpha-1} - \delta - \rho}{\gamma} + \frac{1}{2}(1+\gamma)\left(\frac{C_{A,t}}{C_t}\right)^2 \eta\sigma_A^2 \right]\mathrm{d}t + \frac{C_{A,t}}{C_t}\sqrt{\eta\sigma_A^2}\,\mathrm{d}B_{A,t}, \quad (37)$$

where $C_{A,t} \equiv \partial C\left(K_t, A_t; \eta\right)/\partial A_t$, together with (34) and (35). The deterministic steady-state values of $\overline{A}$, $\overline{K}$, and $\overline{C}$ are given by (21), (22), and (23), respectively.

Alternatively, the equilibrium of the economy can be characterized in the state-space domain by the PDE

$$\left(\alpha\exp(A)K^{\alpha-1} - \delta - \rho\right)C/\gamma + \tfrac{1}{2}(1+\gamma)C\left(C_A/C\right)^2\eta\sigma_A^2 - C_K\left(\exp(A)K^{\alpha} - C - \delta K\right) + C_A\rho_A A - \tfrac{1}{2}C_{AA}\eta\sigma_A^2 = 0, \quad (38)$$

which defines our functional operator $\mathcal{H}\left(C, C_K, C_A, C_{KK}, C_{AA}, C_{KA}, C_{AK}, K, A; \eta\right) = 0$, with unknown solution $C = C\left(K, A; \eta\right)$. Substituting the policy function into (38) yields the functional equation

$$F\left(K, A; \eta\right) \equiv \mathcal{H}\left(C\left(K, A; \eta\right), C_K\left(K, A; \eta\right), C_A\left(K, A; \eta\right), C_{AA}\left(K, A; \eta\right), K, A; \eta\right) = 0, \quad (39)$$

where we have already used the fact that in equilibrium $C_{KK} = C_{KA} = C_{AK} = 0$ since the capital stock is not affected directly by any exogenous shocks.

Let us first consider a first-order perturbation to the unknown policy function around the deterministic steady state

$$C\left(K,A;\eta\right)\approx\overline{C}+\overline{C}_K\left(K-\overline{K}\right)+\overline{C}_A\left(A-\overline{A}\right)+\overline{C}_\eta\eta, \tag{40}$$

where $\overline{C_i}=C_i\left(\overline{K},\overline{A};0\right)$ denotes partial derivative of the policy function with respect to i-th state evaluated at the deterministic steady state. The constants $\overline{C}_K$ and $\overline{C}_A$ are the solution to the quadratic system of equations formed by $F_K(\overline{K},\overline{A};0)=F_A(\overline{K},\overline{A};0)=0$. In particular,

$$\begin{aligned}\overline{C}_K &= \tfrac{1}{2}(\alpha\exp(\overline{A})\overline{K}^{\alpha-1}-\delta)\pm\left[\tfrac{1}{4}(\alpha\exp(\overline{A})\overline{K}^{\alpha-1}-\delta)^2-\alpha\left(\alpha-1\right)\exp(\overline{A})\overline{K}^{\alpha-2}\overline{C}/\gamma\right]^{\frac{1}{2}},\\ \overline{C}_A &= \left(\overline{C}_K+\rho_A\right)^{-1}\left[\overline{C}_K\exp(\overline{A})\overline{K}^{\alpha}-\alpha\exp(\overline{A})\overline{K}^{\alpha-1}\overline{C}/\gamma\right].\end{aligned}$$

For the stochastic growth model we pick the positive root, $\overline{C}_K>0$, since it is the one that is consistent with a concave value function (see Parra-Alvarez, 2018). The remaining constant, $\overline{C}_\eta$, corresponds to the solution of the linear equation $F_\eta(\overline{K},\overline{A};0)=0$, which is given by

$$\overline{C}_\eta=-\left(\overline{C}_K\right)^{-1}\left[\tfrac{1}{2}\left(1+\gamma\right)\overline{C}\left(\overline{C}_A/\overline{C}\right)^2-\tfrac{1}{2}\overline{C}_{AA}\right]\sigma_A^2. \tag{41}$$

Hence, (41) shows that $\overline{C}_\eta\neq 0$, suggesting that a first-order perturbation approximation to the optimal consumption function includes an adjustment for risk. Given the concavity of the consumption function, $\overline{C}_\eta$ is negative. This reflects the fact that risk averse agents will consume less in the presence of risk due to precautionary savings. The risk-correction term requires information on both the slope and the curvature of the optimal consumption function at the deterministic steady state. While $\overline{C}_K$ and $\overline{C}_A$ are already available, we still need $\overline{C}_{AA}$, which results from the solution to the linear system of equations formed by $F_{KK}(\overline{K},\overline{A};0)=F_{KA}(\overline{K},\overline{A};0)=F_{AA}(\overline{K},\overline{A};0)=0$[7].

3.3 An intuition: Why does certainty equivalence break?

The solution to stochastic economic models is said to be certainty equivalent if the resulting policy functions are invariant to higher order moments of the model's underlying exogenous shocks. In other words, the solution of an economic model under uncertainty is identical to the solution of the same model under certainty.

For discrete-time stochastic models certainty equivalence holds for any first-order (linear) approximation around the deterministic steady state. In general, the optimality conditions that characterize economic equilibria in these models can be summarized by a system of stochastic difference equations, where expectations regarding the future value of the control variables need to be formed. Given that the policy functions are *a*

[7] Certainty equivalence will still hold, nonetheless, under the following assumptions on the stochastic growth model: (i) zero risk, $\sigma_A=0$; (ii) quadratic utility, $(1+\gamma)=0$ and $C_{AA}=0$ (see Judd, 1996).

priori unknown, the computation of such expectations can only be done *ex-post* once the optimal controls have been approximated. Hence, for a first-order perturbation this is equivalent to calculating the expected value of a set of linear functions which, according to the linearity property of the expectation operator, implies that only first-order moments will enter the approximated solution.

However, as exemplified by equation (41), this is not the case for continuous-time stochastic models. To illustrate this point consider the non-linear HJB equation in (36), where the expected continuation value $(1/\mathrm{d}t)\mathbb{E}_t[\mathrm{d}V]$ has been already computed using the tools from stochastic calculus. Having computed expectations in the HJB, the resulting Euler equation in (37) includes some features that account for the model's underlying risk. In particular, let us consider the quadratic term $\frac{1}{2}(1+\gamma)C\,(C_A/C)^2\,\eta\sigma_A^2$, which also appears in the correction in (41). The first thing to note is that it contains the marginal response of optimal consumption to changes in the exogenous driving force of the model, C_A, which is closely related to risk aversion. To see this recall that in equilibrium the optimal consumption function, C, is related to the marginal utility of consumption, $u'(C)$, and thus C_A is related to the first-order derivative of the marginal utility, $u''(C)$. Also note that it contains the perturbation and the variance parameters which jointly capture the amount of risk in the model, $\eta\sigma_A^2$. Finally, note that the term $1+\gamma$ can be shown to be the coefficient of relative prudence for the case of CRRA utility functions. In contrast, as expected values cannot be computed *a priori* for discrete-time models, the Euler equation for consumption in that case will only include terms related to the marginal utility of consumption, $u'(C)$. If, in contrast, we choose the standard deviation of the shock to be the perturbation parameter, then it is easy to show that the drift of the Euler equation in (37) depends instead on the term $\eta^2\sigma_A^2$. Thus, the risk correction in (41) will be a function of η which implies $\overline{C}_\eta = 0$, and thus certainty equivalence, when evaluated at the deterministic steady state.

How this relates to certainty equivalence becomes clear when taking a closer look at the precautionary motive, or prudence, that describes the optimal reaction of consumption to risk. Prudence is related to the third derivative of the utility function, $u'''(C)$[8], and its absence leads to certainty equivalence. Hence, a policy function that only contains $u''(C)$ will account for risk aversion, i.e., how much an agent dislikes risk, but not for prudence and, thus, will be certainty equivalent. If, in addition, the policy function involves the fourth derivative of the utility function, $u^{(4)}(C) < 0$, then it will also account for temperance, i.e., how the marginal propensity to consume responds to risk. Thus, while the effects of risk on the level of consumption are captured by $u'''(C)$, the effects on the slope are captured by $u^{(4)}(C)$ (see Kimball, 1990a and Zeldes, 1989).

In terms of the approximation method note that a first-order (linear) perturbation to

[8] Absolute prudence is defined as $-u'''(C)/u''(C)$, while relative prudence is defined as $-u'''(C)\,C/u''(C)$ (see Kimball, 1990b).

Risk effects on:	$\partial^n u/(\partial c)^n$	related to:	Cont. Time		Discrete Time		
			1st	2nd	1st	2nd	3rd
—	$n = 2$	risk aversion: $-u''/u'$	✓	✓	✓	✓	✓
level of C	$n = 3$	prudence: $-u'''/u''$	✓	✓		✓	✓
slope of C	$n = 4$	temperance: $u^{(4)} < 0$		✓			✓

Table 1. Effects of risk in perturbation solutions. The table indicates the order of the derivative of the utility function $\partial^n u/(\partial c)^n$ necessary to account for a particular effect of risk on optimal consumption, as well as the order of approximation needed to capture it both in continuous-time and discrete-time stochastic models.

the unknown consumption function requires computing the first derivative of the Euler equation. Since its discrete-time version only contains $u'(C)$, a first-order approximation will just include terms up to the second derivative of the utility function and hence it will account for risk aversion but not for prudence. Therefore, a first-order approximation in discrete time will be certainty equivalent. In contrast, the continuous-time Euler equation (37) already includes terms related to $u'(C)$ and $u''(C)$, so its first-order approximation will account for both risk aversion *and* prudence. The resulting policy functions in continuous time will not only depend on the mean of the exogenous shock but also on its variance – *breaking certainty equivalence*. To break certainty equivalence in discrete time, a second-order approximation is needed, which in continuous time already leads to correction terms in the slopes. Table 1 summarizes the discussion above by indicating which order of approximation is required in order to account for a given risk effect.

3.4 Risky steady state

Similar to the concept of the deterministic steady state, we may define the risky or stochastic steady state as the fixed point to which the dynamic economic system converges to in the absence of shocks, but where $\sigma_A > 0$. As discussed in Coeurdacier et al. (2011), the risky steady state is of utmost relevance to the extent that it incorporates relevant information regarding the future risk prospects of risk-averse economic agents.

Unfortunately, the computation of the risky steady state is not straightforward. Following its definition, we require information about how risk, as measured by the variance of economic shocks, affects the policy functions, $\mathbf{g}(\mathbf{x}; \eta)$, which, *ex-ante*, are also unknown. However, it is still possible to approximate its value by using the perturbation-based approximation of the policy functions around the deterministic steady state.

In particular, we define the risky steady state value of the state variables, $\hat{\mathbf{x}}$, as the solution to the system of (non-linear) equations formed by

$$\mathbf{f}(\hat{\mathbf{x}}, \mathbf{g}(\hat{\mathbf{x}}; \eta = 1)) = \mathbf{0}, \tag{42}$$

where $\mathbf{f}(\cdot)$ is the drift in (28) that results from (i) replacing the vector of controls $\mathbf{y}$ by its perturbation-based approximation evaluated at the unknown risky steady state; (ii) setting any future realization of economic shocks to zero, i.e. $\mathrm{d}\mathbf{B}_t = \mathbf{0}$; and (iii) imposing the stationarity condition $\mathrm{d}\mathbf{x}_t/\mathrm{d}t = \mathbf{0}$. Once $\hat{\mathbf{x}}$ is computed, it is possible to compute the risky steady state value for the control vector as $\hat{\mathbf{y}} = \mathbf{g}(\hat{\mathbf{x}}; \eta = 1)$.

Since it is already possible to account for risk in continuous time using a first-order approximation, the approach in (42) can be used to build an approximation to the risky steady state in a linear framework[9]. Hence, the first-order approximation of the risky steady state is given by the solution to

$$\mathbf{0} = \mathbf{f}\left(\hat{\mathbf{x}}, \mathbf{g}(\bar{\mathbf{x}}; 0) + \mathbf{g}_{\mathbf{x}}(\bar{\mathbf{x}}; 0)(\hat{\mathbf{x}} - \bar{\mathbf{x}}) + \mathbf{g}_{\eta}(\bar{\mathbf{x}}; 0)\right) \tag{43}$$

$$\hat{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}; 0) + \mathbf{g}_{\mathbf{x}}(\bar{\mathbf{x}}; 0)(\hat{\mathbf{x}} - \bar{\mathbf{x}}) + \mathbf{g}_{\eta}(\bar{\mathbf{x}}; 0). \tag{44}$$

A similar procedure can be used to build a k-th order approximation of the risky steady state, for $k > 1$.

4 Asset pricing

This section investigates the economic implications of the approximated solutions by measuring the pricing errors made when using the First-Order CE approximation, the First-Order approximation, and the Second-Order approximation defined in Section 3.1. The pricing mismatch is computed relative to a benchmark that is obtained using a global non-linear projection method based on a Chebyshev polynomial approximation of the unknown value function (see Parra-Alvarez, 2018; Posch, 2018). This approach delivers highly accurate solutions but is costly in terms of computational efficiency. By comparing the different pricing errors we can study how risk matters quantitatively for asset pricing.

4.1 Stochastic discount factor

We define the stochastic discount factor (SDF) as the process m_s/m_t , such that, for any security with price P_t, and a single payoff χ_s at some future date $s \geq t$, we obtain

$$m_t P_t = \mathbb{E}_t\left[m_s \chi_s\right] \quad \Rightarrow \quad 1 = \mathbb{E}_t\left[(m_s/m_t) R_s\right], \tag{45}$$

where $R_s \equiv \chi_s/P_t$ is the security's return, and m_t is the present (discounted) value of a unit of consumption in period t. Hence, the condition (45) can be used to discount expected payoffs on *any* asset with a single payoff to find their equilibrium prices. In

[9] A similar methodology is available for the case of discrete-time models for perturbations of order 2 and higher (see de Groot, 2013). A summary of this procedure is described in the accompanying web appendix.

other words, investors will be indifferent between investing into the various assets only if (45) is satisfied.

From the definition of expected discounted life-time utility (1), the instantaneous utility function (2), and the first-order condition (11), we obtain the SDF for $s > t$ following Detemple and Zapatero (1991) as (see Appendix A)

$$m_s/m_t = e^{-\rho(s-t)} \frac{(C_s - X_s)^{-\gamma} + bV_{X,s}}{(C_t - X_t)^{-\gamma} + bV_{X,t}} \quad \text{for} \quad s > t, \tag{46}$$

where $m_t = e^{-\rho t}(C_t - X_t)^{-\gamma} + bV_{X,t}$ is the present discounted value of a unit of consumption at instant $t \geq 0$.

4.2 Pricing errors

In what follows we define pricing errors as (see Lettau and Ludvigson, 2009)

$$\varepsilon_i \equiv \mathbb{E}_t[(m_s/m_t)R_{i,s}] - 1, \tag{47}$$

based on the gross return on any tradable asset i with instantaneous return, $R_{i,s}$.

Risk-free asset. Consider a zero-coupon bond with sure payoff $\chi_{f,t+N} = 1$ at period $t+N$. From (45) we obtain the price of this zero-coupon bond as $P_{f,t}^{(N)} = \mathbb{E}_t\left[(m_{t+N}/m_t)\right]$ such that the return of this asset is $R_{f,t}^{(N)} = 1/P_{f,t}^{(N)}$, conditional on the information set at time t. Unfortunately, we do not readily observe an *instantaneous* risk-free asset for $N \to 0$ with corresponding yield $r_t^f \equiv \lim_{N\to 0} R_{f,t}^{(N)}$. Any equilibrium return from a risk-free sovereign bond carries a term premium for a given time-to-maturity $N \equiv s - t$.

Zero-coupon bonds. To compute the price of a zero-coupon bond for a given time-to-maturity N we use the partial differential equation (PDE) approach (see Posch, 2018). Hence, in the absence of arbitrage opportunities, the fundamental price of a zero-coupon bond with maturity N, $P_{f,t}^{(N)}$, satisfies

$$\mathbb{E}_t\left(\frac{\mathrm{d}P_{f,t}^{(N)}}{P_{f,t}^{(N)}}\right) - \left(\frac{1}{P_{f,t}^{(N)}}\frac{\partial P_{f,t}^{(N)}}{\partial N} + r_t^f\right)\mathrm{d}t = -\mathbb{E}_t\left(\frac{\mathrm{d}P_{f,t}^{(N)}}{P_{f,t}^{(N)}}\frac{\mathrm{d}m_t}{m_t}\right), \tag{48}$$

with boundary condition $P_{f,t}^{(0)} = 1$, and where the SDF evolves according to

$$\frac{\mathrm{d}m_t}{m_t} \equiv \mu_{m,t}\mathrm{d}t + \sigma_{m,t}\mathrm{d}B_{A,t}, \tag{49}$$

where the drift, $\mu_{m,t}$, and diffusion, $\sigma_{m,t}$, terms in (49) are functions of the policy functions. Hence, they depend on the approximation method used (see Appendix A).

Assuming that the market price yields the efficient price under the physical probability measure $\mathbb{P}$, the fundamental pricing equation can be written as

$$\frac{\partial P_{f,t}^{(N)}}{\partial N} = \frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{d}m_t}{m_t}\right) P_{f,t}^{(N)} + \frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left(\mathrm{d}P_{f,t}^{(N)}\right) + \frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left((\mathrm{d}P_{f,t}^{(N)})\left(\frac{\mathrm{d}m_t}{m_t}\right)\right),$$

where we used the fact that $r_t^f = -\frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}[\mathrm{d}m_t/m_t]$. The physical probability measure is defined in terms of the (true) policy function for consumption from the economic model. In what follows, we assume that the latter is approximated accurately (in a global sense) by means of projection methods. Note that the covariance of the prices with the SDF on the right-hand side of (48) gives rise to a term premium.

Since in equilibrium all the time dependence of the zero-coupon bond price with given time-to-maturity N comes through the state variables that drive the economy, i.e., $P_{f,t}^{(N)} = P_f^{(N)}(K_t, X_t, A_t)$, an application of Itô's lemma shows that the dynamics of the bond's price is given by

$$\mathrm{d}P_{f,t}^{(N)} = \frac{\partial P_{f,t}^{(N)}}{\partial K_t}\mathrm{d}K_t + \frac{\partial P_{f,t}^{(N)}}{\partial X_t}\mathrm{d}X_t + \frac{\partial P_{f,t}^{(N)}}{\partial A_t}\mathrm{d}A_t + \frac{1}{2}\frac{\partial^2 P_{f,t}^{(N)}}{\partial A_t^2}(\mathrm{d}A_t)^2. \quad (50)$$

Hence, the fundamental pricing equation can now be written as

$$\begin{aligned} \frac{\partial P_{f,t}^{(N)}}{\partial N} &= \mu_{m,t}P_{f,t}^{(N)} + \frac{\partial P_{f,t}^{(N)}}{\partial K_t}(\Phi((\exp(A_t)K_t^\alpha - C(K_t, X_t, A_t))/K_t) - \delta)K_t \\ &+ \frac{\partial P_{f,t}^{(N)}}{\partial X_t}(bC(K_t, X_t, A_t) - aX_t) - \rho_A A_t \frac{\partial P_{f,t}^{(N)}}{\partial A_t} \\ &+ \frac{1}{2}\frac{\partial^2 P_{f,t}^{(N)}}{\partial A_t^2}\sigma_A^2 + \frac{\partial P_{f,t}^{(N)}}{\partial A_t}\sigma_A\sigma_{m,t}. \end{aligned} \quad (51)$$

The functional form of the solution to (51) is unknown. We use collocation methods to approximate the price function with the polynomial $P_{f,t}^{(N)} \approx \boldsymbol{\phi}(N, K_t, X_t, A_t)\mathbf{c}$, in which $\mathbf{c}$ is a vector of unknown coefficients and $\boldsymbol{\phi}(\cdot)$ denotes the Chebyshev basis matrix with associated Chebyshev nodes. We extend the approximated PDE in (51) with the boundary condition $\boldsymbol{\phi}(0, K_t, X_t, A_t)\mathbf{c} = \mathbf{1}_p$, where p denotes the degree of the approximation. The collocation approach provides accurate results and allows us to avoid tedious numerical simulations.

Let $\varepsilon_{f,a}^{(N)}$ represent a measure of *ex-ante* pricing errors on a zero-coupon bond with time-to-maturity N, defined as the (absolute) percentage deviation of the price under the subjective probability measure $\mathbb{S}$ relative to the physical probability measure $\mathbb{P}$

$$\varepsilon_{f,a}^{(N)} \equiv \mathbb{E}_t^{\mathbb{S}}[(m_N/m_t)]/\mathbb{E}_t^{\mathbb{P}}[(m_N/m_t)] - 1. \quad (52)$$

We define $\mathbb{S}$ to be the probability measure used by an investor that uses the perturbation approach to approximate the model's policy function for consumption instead of the true solution (i.e. it employs either the First-Order CE, the First-Order, or the Second-Order approximation) in both the SDF dynamics in (49) and the bond-price PDE in (50). Let $P_{f,a}^{(N)}$ denote the price that solves (51) under the subjective probability measure $\mathbb{S}$.

By using the PDE approach it is possible to shed light on the sources of pricing errors. First, the risk-free rate is poorly approximated. Second, the covariance of the price dynamics with the SDF is poorly captured. And third, the approximation to the consumption function alone is inaccurate. Hence, we can decompose the pricing errors into misspecification of: (i) the risk-free rate (or drift of the SDF), $r_t^f = -\mu_{m,t}$; (ii) the term premium that arises from the covariance component, $(\partial P_{f,t}^{(N)}/\partial A_t)\sigma_A \sigma_{m,t}$; and (iii) the price dynamics $\mathbb{E}_t(\mathrm{d}P_{f,t}^{(N)})/P_{f,t}^{(N)}$.

To analyze the different sources of pricing mismatch, we define the *ex-post* pricing errors as the (absolute) percentage deviation of the price under the subjective probability measure $\mathbb{S}$ relative to the physical measure $\mathbb{P}$, but where the investor can observe the correct/true SDF dynamics either partially

$$\varepsilon_{f,b}^{(N)} \equiv \mathbb{E}_t^{\mathbb{S}}\left[(m_N/m_t)|\mu_{m,t}^{\mathbb{S}} \equiv \mu_{m,t}\right]/\mathbb{E}_t^{\mathbb{P}}[(m_N/m_t)] - 1, \tag{53}$$

or completely

$$\varepsilon_{f,c}^{(N)} \equiv \mathbb{E}_t^{\mathbb{S}}\left[(m_N/m_t)|\mu_{m,t}^{\mathbb{S}} \equiv \mu_{m,t}, \sigma_{m,t}^{\mathbb{S}} \equiv \sigma_{m,t}\right]/\mathbb{E}_t^{\mathbb{P}}[(m_N/m_t)] - 1. \tag{54}$$

Hence, the measures in (53) and (54) focus on the pricing error reduction obtained by providing further information on the SDF dynamics. For example in (54), the investor infers the correct SDF from the data, and solves the corresponding PDE

$$\frac{\partial P_{f,c}^{(N)}}{\partial N} = \mu_{m,t} P_{f,c}^{(N)} + \frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{S}}\left(\mathrm{d}P_{f,c}^{(N)}\right) + \left(\frac{\partial P_{f,c}^{(N)}}{\partial A_t}\right) P_{f,c}^{(N)} \sigma_A \sigma_{m,t},$$

with the approximated price $P_{f,c}^{(N)}$ (in the same way we define $P_{f,b}^{(N)}$). This enables us to study the hypothetical error an investor would face ex-post when trading the asset at the subjective (approximated) price instead of true $P_{f,t}^{(N)}$, yet knowing the SDF dynamics.

5 Results

5.1 Calibration

To quantitatively evaluate the extent to which the First-Order approximation can account for the effects of risk we proceed to calibrate the prototype model of Section 2 to an annual frequency. Therefore, all the parameter values should be interpreted ac-

Parameter	Value	Source / Target
Discounting, ρ	0.0410	Jermann (1998)
Risk aversion, γ	2.0000	Aruoba et al. (2006)
Depreciation rate, δ	0.0963	Jermann (1998)
Capital share in output, α	0.3600	Jermann (1998)
Persistence TFP, ρ_A	0.2052	Aruoba et al. (2006)
Volatility TFP, σ_A	0.0307	U.S. real GDP growth volatility
Adjustment cost, ξ	0.3261	Short-term return on government bonds reported in Jermann (1998)
Habit current cons., b	0.8200	Jermann (1998)
Habit past cons., a	1.0000	Jermann (1998)

Table 2. Parameter values. The parameters of the model are calibrated to an annual frequency and their values should be interpreted accordingly.

cordingly. Many of the parameter values are chosen to replicate the parameterization to the U.S. economy used in the discrete-time models of Jermann (1998) and Aruoba et al. (2006). A complete summary of the model's calibration is provided in Table 2.

In particular, we set the risk aversion parameter and the share of capital income to $\gamma = 2$ and $\alpha = 0.36$, respectively. The values for the subjective discount rate, the depreciation rate and the habit process are set to $\rho = 0.041$, $\delta = 0.0963$, and $a = 1$ and $b = 0.82$, respectively. These parameter values are consistent with steady-state values for the capital-output ratio, and the consumption and investment shares in aggregate output of around 2.5, and 76% and 24%, respectively. We fix the adjustment cost parameter to $\xi = 0.3261$ such that the model produces an average real return on short term government bonds close to that reported in Jermann (1998). Finally, the persistence of TFP is set to $\rho_A = 0.2052$ which corresponds to the continuously compounded value of that in Aruoba et al. (2006), while its volatility is set to $\sigma_A = 0.0307$ to target the relative growth volatilities (relative standard deviations) of consumption and investment to output, and which is consistent with the observed volatility of real GDP growth in the U.S. for the period 1954-1989.

Table 3 reports some of the moments implied by different parameterizations of our RBC model when solved by a first-order perturbation and a global approximation method based on collocations. Along with the prototype model of Section 2 (Benchmark), we report the moments for the model without habit formation and no capital adjustment cost of Section 3.2 (No habits, no adj. costs, i.e. $b = X_0 = 0$ and $\xi \to \infty$), no capital adjustment cost (Habit, no adjustment costs, i.e, $\xi \to \infty$), and without habits (Adjustment costs, no habits, i.e., $b = X_0 = 0$). The last row in the table shows the moments reported by Jermann (1998) for U.S. data from 1954-1989. The relative standard deviations for quarterly consumption and investment growth correspond to averages over 100,000 samples generated through a Euler-Maruyama discretization scheme with precision $\Delta = 0.0125$, each of them consisting of 10 years of simulated data, initialized at the deterministic steady

Model version	σ_C/σ_Y		σ_I/σ_Y		$R_{f,t}^{(0.25)}$	
	Pert.	Global	Pert.	Global	Pert.	Global
Benchmark	0.45	0.44	2.65	2.69	0.37 (4.99)	0.68 (5.17)
No habits, no adj. costs	0.34	0.34	3.00	3.01	4.09 (0.19)	4.11 (0.19)
Habit, no adj. costs	0.13	0.14	3.72	3.72	3.86 (0.26)	4.10 (0.19)
Adj. costs, no habits	1.12	1.11	0.68	0.66	3.77 (0.60)	3.85 (0.61)
U.S. Data (1954-1989)	0.51		2.65		0.80 (5.67)	

Table 3. Moments from simulated data. The different moments are computed using 100,000 draws starting at the deterministic steady state. The policy functions are computed using both a first-order perturbation (Pert.) and a global method (Global). For comparison, U.S. data moments correspond to those in Jermann (1998). We report the standard deviation (sd) of quarterly growth rates for output, σ_Y, consumption, σ_C, and investment, σ_I after 10 years; and the three-month yield $R_{f,t}^{(0.25)}$ with sd in brackets (annualized, percentage terms).

state. Finally, the table also includes the three-month simulated yield-to-maturity for a zero-coupon bond and the standard deviation of its simulated distribution.

We confirm that only the model with both habit formation and capital adjustment costs generates the historical consumption and investment volatility relative to output, and three-month bond yields with sufficient variability. Hence, in this model risk matters quantitatively and we can use the parameterization in Table 2 to investigate the asset pricing errors for the different solution methods (at the deterministic steady state).

5.2 Approximated policy functions

Figure 1 shows the first- and second-order perturbation approximations to the policy function for consumption around the deterministic steady state for our prototype model using the calibration in Table 2. The left panel shows optimal consumption along the capital stock lattice for values 15% below and above its deterministic steady state, while keeping the remaining state variables fixed at their deterministic steady state values. The right panel plots optimal consumption along the habit formation lattice covering values that are 15% above and below its deterministic steady state value. The figures also indicate the deterministic and risky steady state values for consumption, capital stock and habit. Their values are reported in Table 4, where we have also included a measure of the risky steady state computed from a global approximation based on projection methods[10].

The plot depicts two types of a first-order (linear) approximation to the optimal consumption function. First, it shows the First-Order (CE) by the dotted line, which resembles the first-order approximation one would obtain from a discrete-time version of the model. By construction, this approximation is invariant to the amount of volatil-

[10] The second-order risky steady-state values for the corresponding discrete-time model approximated with the approach in de Groot (2013) are very close to the second-order risky-steady state values that we report in Table 4.

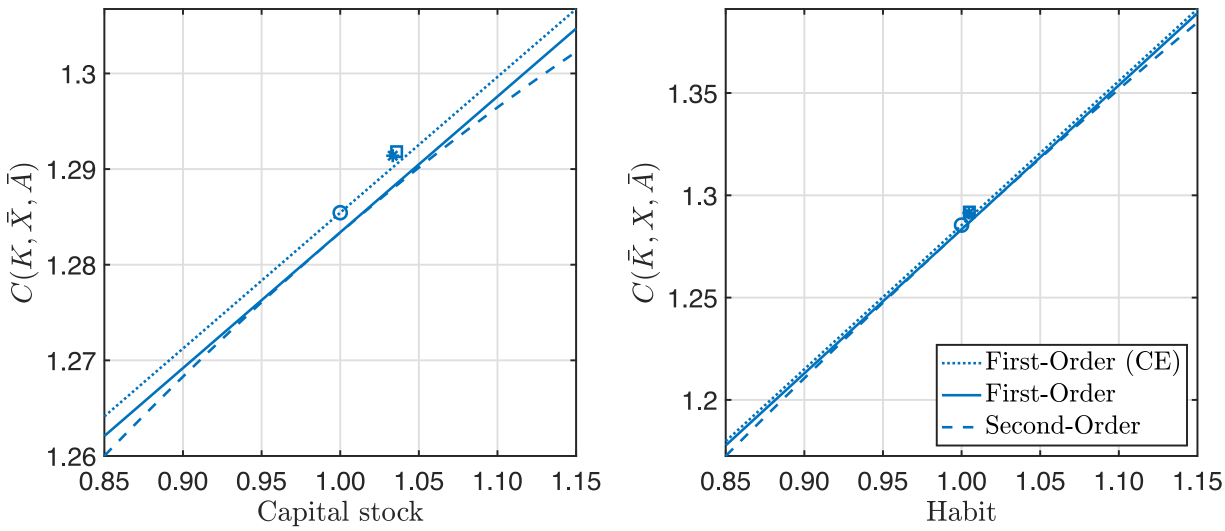

Figure 1. Approximated policy function for consumption: First- and second-order approximations to the policy function for consumption around the deterministic steady state along the capital stock lattice (left panel) and the habit lattice (right panel), while keeping the remaining state variables at their corresponding deterministic steady-state values. Values on the horizontal axis represent deviations from deterministic steady-state values. A circle denotes the deterministic steady state, a star denotes the first-order approximation to the risky steady state, and a square denotes the second-order approximation to the risky steady state.

ity in the model, and hence is certainty equivalent. Second, the solid line depicts the First-Order approximation that corresponds to the first-order perturbation solution that breaks certainty equivalence as it includes the (constant) risk correction term $\overline{C}_\eta \equiv C_\eta\left(\overline{K}, \overline{X}, \overline{A}; 0\right) \neq 0$. Hence, while still being a linear approximate solution, it is risk sensitive as its intercept depends on the amount of uncertainty in the model. As a comparison, we plot the Second-Order approximation (dashed line) to illustrate the additional risk correction attainable when using higher orders of approximation.

Two things are worth mentioning at this point. First, note that the First-Order (CE) policy function for consumption, which by construction passes through the deterministic steady state, lays above the other two alternative approximations. The reason is that the latter account for the effects of risk, and hence imply lower consumption levels along the entire state space. In particular, the First-Order approximation is parallel to the First-Order (CE), and for values of the state space in a neighborhood of the deterministic steady state, it will imply levels of consumption that are relatively close to those suggested by the Second-Order, and hence, a non-linear approximation. Second, the risky steady states computed from the First- and Second-Order approximations command higher values for the capital stock, habits, and consumption over the long-run, relative to those implied by the deterministic case. This result can be confirmed by looking at Table 4. The higher risky steady-state values result from households that consume less and save more in the short run due to precautionary motives and hence imply higher levels of capital stock and consumption over the long run.

Variable	Deterministic	Mean	Risky		
			First-Order	Second-Order	Global
A	0	0	0	0	0
X	1.0541	1.0533	1.0589	1.0593	1.0592
K	4.5077	4.5662	4.6582	4.6693	4.6655
C	1.2854	1.2847	1.2914	1.2918	1.2917

Table 4. Steady states values. The table reports steady-state values for all the variables in the model. It includes the exact deterministic steady-state values, the simulated ergodic mean (global solution), as well as the first- and second-order approximated risky steady-state values. A global approximation to the risky steady state is included as a benchmark.

A detailed summary of the approximated policy function for consumption is presented in Table 5, where we report the loadings from the first- and second-order perturbations associated to each of the state variables. Columns 2 and 4 show the coefficients for the continuous-time model, while columns 3 and 5 do the same for its discrete-time version[11]. Comparing the first-order approximations in columns 2 and 3 confirms that we break certainty equivalence when time is continuous. Following our previous discussion, the constant risk correction of -0.0020 implied by our calibration, which is otherwise absent in the solution to the discrete-time model, suggests that a First-Order (CE) approximation overestimates optimal consumption in the presence of uncertainty along the entire state space. Note how a similar risk correction of -0.0025 is obtained in a discrete-time framework when using a second-order, and hence non-linear, approximation. Comparing columns 4 and 5 reveals that a second-order perturbation in continuous-time includes not only an additional adjustment in the constant term of the approximation, $\overline{C}_{\eta\eta} \neq 0$, but also in the slopes of the policy function implying a time-varying risk correction. As suggested in Andreasen (2012), these two additional effects can only be achieved in discrete-time models by computing third-order approximations (see Table 1).

5.3 Impulse response functions

Having approximated the unknown policy function, we now compute the impulse-response functions (IRF) in order to compare how the different degrees of approximation capture the amplification and propagation mechanisms of the prototype economy to a temporary shock on the level of TFP. The results are presented in Figure 2, where we plot the transitional dynamics of consumption, capital stock, habits, and output over the course of 60 years after a one-time unexpected increase in TFP equal to σ_A. Prior to the shock, all the variables are assumed equal to their respective stationary values. Thus, while the First-Order (CE) solution is initially resting at the deterministic steady state, the First-

[11]The first- and second-order approximations to the policy functions that solve the corresponding discrete-time model are computed using `Dynare`.

	First-Order		Second-Order	
	Cont. time	Disc. time	Cont. time	Disc. time
$\overline{C}$	1.2854	1.2854	1.2854	1.2854
$\overline{C}_\eta$	$\mathbf{-0.0020}$	**0**	-0.0020	0
$(K-\overline{K})$	0.0315	0.0290	0.0315	0.0290
$(X-\overline{X})$	0.6680	0.7042	0.6680	0.7042
$(A-\overline{A})$	0.5370	0.4899	0.5370	0.4899
$\overline{C}_{\eta\eta}$	-	-	$\mathbf{-0.0000}$	$\mathbf{-0.0025}$
$(K-\overline{K})\times\eta$	-	-	$\mathbf{-0.0003}$	**0**
$(X-\overline{X})\times\eta$	-	-	**0.0020**	**0**
$(A-\overline{A})\times\eta$	-	-	$\mathbf{-0.0063}$	**0**
$(K-\overline{K})^2$	-	-	-0.0049	-0.0046
$(X-\overline{X})^2$	-	-	-0.1930	-0.2089
$(A-\overline{A})^2$	-	-	-0.3119	-0.3663
$(K-\overline{K})\times(X-\overline{X})$	-	-	0.0402	0.0389
$(K-\overline{K})\times(A-\overline{A})$	-	-	-0.0282	-0.0286
$(A-\overline{A})\times(X-\overline{X})$	-	-	0.6508	0.6942

Table 5. Loadings of policy function for consumption. The table reports the coefficients from first- and second-order approximations to the policy function for consumption, $C = C(K, X, A; \eta)$, around the deterministic steady state $(\overline{C}, \overline{K}, \overline{A})$ for the model in Section 2 and its equivalent discrete-time version.

and Second-Order solutions are resting at their respective (approximate) risky steady states. Again, the CE solution can be thought of as a proxy for the IRF one would obtain from a first-order perturbation to an equivalent discrete-time model. For comparison purposes we report the first- and second-order IRFs for the discrete-time model in the accompanying web appendix.

Note that the IRFs for the First-Order (CE) lay below the risk-sensitive approximations in Figure 2. Intuitively, since the constant correction term for the First-Order approximation is negative, $\overline{C}_\eta < 0$, one may expect that the consumption response approximated by a First-Order will be below the one approximated by First-Order (CE). However, as shown in Figure 1 and Table 4, the risk-correction reduction in consumption induced by the former will in fact lead to a higher risky steady-state capital stock and, thereby, a higher risky steady-state level of consumption. Thus, the fact that the First-Order (CE) is below the First-Order and Second-Order responses is explained by the differences in their fixed points, or long-run convergence levels, hence cannot readily interpreted as an indication that certainty equivalent approximations underestimate the response of macroeconomic variables to aggregate shocks. Furthermore, note that the additional risk-corrections provided by the Second-Order approximation (and hence non-linear) have only minor effects on the optimal reaction of consumption to a TFP shock. In other words, the risk-correction in the first-order perturbation approach provides a

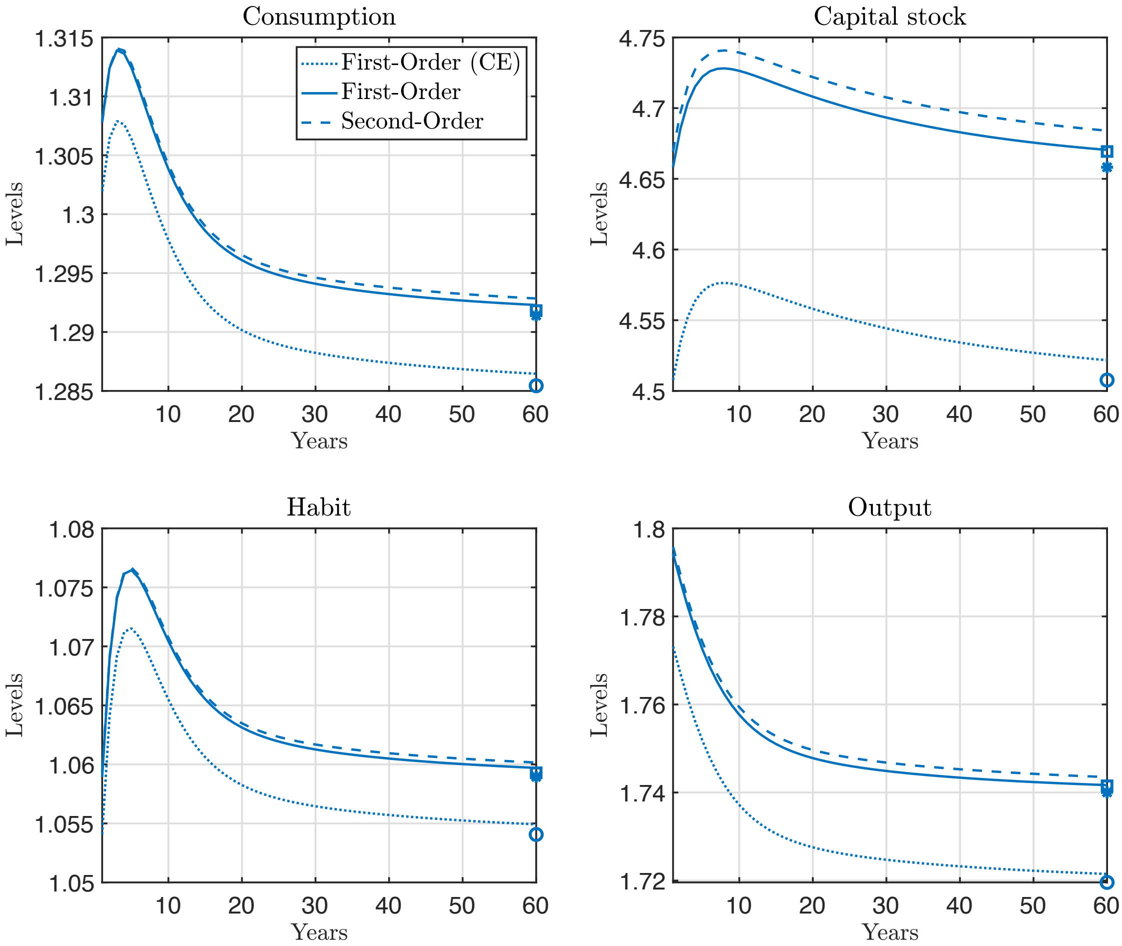

Figure 2. Impulse-Response function to a one s.d. shock in TFP: Responses for the annual levels of consumption, capital stock, habit formation and output to a one time shock in TFP equivalent to one standard deviation, σ_A. All the variables are assumed to be in their corresponding steady states before the shock. A circle denotes the deterministic steady state, a star denotes the first-order approximation to the risky steady state, and a square denotes the second-order approximation to the risky steady state.

sensible approximation to the effects of risk in continuous-time models.

5.4 Asset pricing implications

In this section we investigate the ability of the different approximations to account for risk when pricing assets. While others focus on accuracy measures based on the computation of Euler equation errors (cf. Judd, 1998; Aruoba et al., 2006; Parra-Alvarez, 2018), we are more interested in the implications that each of the approximations have on the pricing mismatch incurred by an investor that does not have/use the 'true' solution of the model.

In what follows, we use the PDE approach introduced in Section 4 to assess to what extent our first-order approximation can account for risk. Table 6 reports the absolute pricing errors of a zero-coupon bond for different time-to-maturities when the economy is at its deterministic steady state, $\varepsilon_f^{(N)} \equiv \varepsilon_f^{(N)}\left(\overline{K}, \overline{X}, \overline{A}\right)$. We show the results for risk-free

	Global	First-Order (CE)	First-Order	Second-Order
$\varepsilon_{f,a}^{(0.25)}$	0.0000	**0.0095**	**0.0014**	**0.0003**
$\varepsilon_{f,b}^{(0.25)}$		0.0001	0.0001	0.0000
$\varepsilon_{f,c}^{(0.25)}$		0.0001	0.0000	0.0000
$\varepsilon_{f,a}^{(1)}$	0.0000	0.0494	0.0065	0.0009
$\varepsilon_{f,b}^{(1)}$		**0.0017**	**0.0009**	**0.0001**
$\varepsilon_{f,c}^{(1)}$		0.0018	0.0000	0.0000
$\varepsilon_{f,a}^{(5)}$	0.0000	0.4269	0.1742	0.0050
$\varepsilon_{f,b}^{(5)}$		0.0243	0.0096	0.0069
$\varepsilon_{f,c}^{(5)}$		**0.0302**	**0.0017**	**0.0000**

Table 6. Asset pricing implications for zero-coupon bonds. The table reports the absolute pricing errors induced by different approximation methods on risk-free zero-coupon bonds with time-to-maturity of three months, one year and five years when the SDF dynamics are (a) are not observed, (b) partially observed (drift only) and (c) completely observed (drift and diffusion).

bonds with a 3-month, 1 year and 5 years time-to-maturity. Column 2 (Global) provides errors under the true probability measure $\mathbb{P}$ as obtained from a global approximation using collocation methods; columns $3-5$ report errors resulting from First-Order (CE), First-Order, and Second-Order perturbation. Moreover, for each time-to-maturity, we report errors for the cases in which the SDF dynamics are (a) not observed as in (52), (b) partially observed (drift only) as in (53), or (c) fully observed (drift and diffusion) as in (54). This means that in (a) we approximate the policy function as well as the drift and diffusion of the SDF, in (b) we approximate the policy function and diffusion of the SDF, and finally in (c) we only approximate the policy function.

Our results suggest that there are substantial gains from using the (risk-sensitive) First-Order relative to the First-Order (CE) version. Consider the case of a three-month zero-coupon bond with effective price of $P_f^{0.25} = 0.9983$. An investor relying on the certainty equivalent linear solution (First-Order (CE)), not just to approximate the policy function but also the drift and diffusion of the SDF, will incur in pricing errors of about 1 dollar for each 100 dollar spent ($\varepsilon_{f,a}^{(0.25)} = 0.95\%$ in Column 3). If instead the investor uses the First-Order approximation, the pricing error will be of the order of 10 cents for each 100 dollars spent ($\varepsilon_{f,a}^{(0.25)} = 0.14\%$ in Column 4). Hence, breaking certainty equivalence reduces the potential price mismatch by nearly 90% while still remaining in the linear world. The Second-Order approximation further reduces pricing errors which fall to about 3 cents per 100 dollars ($\varepsilon_{f,a}^{(0.25)} = 0.03\%$ in Column 5). Sizable gains are also observed for bonds with longer time-to-maturities or from inferring the dynamics of the SDF (drift and diffusion) from the data.

We find that although pricing errors increase as N increases, they can be substantially reduced if the investor uses the true risk-free rate (or drift of the SDF), which is

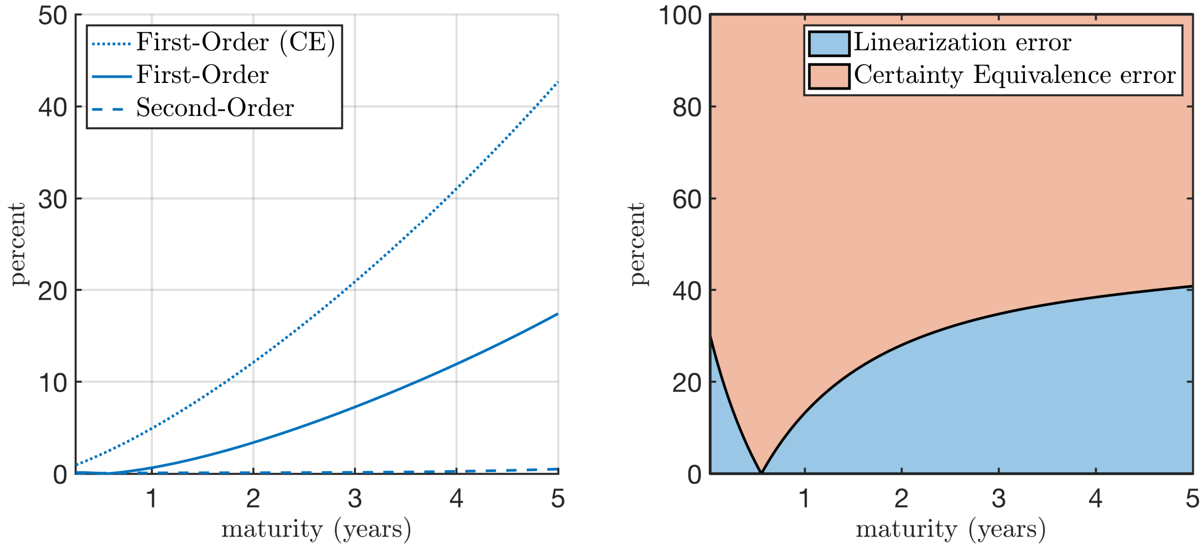

Figure 3. Pricing errors: The left panel plots the absolute pricing errors for different approximations. The right panel decomposes the pricing error incurred by using the CE solution into a certainty equivalence component and a linearization component, where the line represents |(First-Order)/(First-Order (CE)) − 1|.

case (b). In fact, the pricing error for the First-Order approximation is below 10 cents ($\varepsilon_{f,b}^{(1)} = 0.09\%$ in Column 4) for the one year ahead zero-coupon bond. Relative to the First-Order (CE) solution, the risk-adjusted First-Order approximation reduces pricing errors by about 50 percent for one year maturities when using the true risk-free rate. If the investor further knows the true diffusion of the SDF, case (c), i.e., when only the policy functions are approximated, then the First-Order approximation also performs well for a maturities of 5 years. Here, the error is about 20 cents ($\varepsilon_{f,c}^{(5)} = 0.17\%$ in Column 4), which is 90% lower than the error from First-Order (CE) ($\varepsilon_{f,c}^{(5)} = 3.02\%$ in Column 3).

Figure 3 confirms our results for the case of *ex-ante* absolute pricing errors. Investors using a First-Order (CE) solution, when the true data generating process (DGP) is the global solution, would accept large and persistent pricing errors (see Lettau and Ludvigson, 2009): they range between one to five percent for bonds with time-to-maturities from a quarter to a year ($\varepsilon_{f,a}^{(0.25)} = 0.95\%$, $\varepsilon_{f,a}^{(1)} = 4.94\%$). On the contrary, those using the First-Order approximation can reduce these pricing errors by more than 85 percent ($\varepsilon_{f,a}^{(0.25)} = 0.14\%$, $\varepsilon_{f,a}^{(1)} = 0.65\%$).

To better understand the sources of the pricing errors, the right panel of Figure 3 decomposes the pricing mismatch made when using the First-Order (CE) solution into: (i) the error stemming from linearization in the presence of uncertainty; and (ii) the error stemming from imposing certainty equivalence in the linear world. As (i) is given by the error resulting from the First-Order approximation, (ii) results as the (absolute) difference of the errors from the First-Order and the First-Order (CE) approximation, and hence provides a measure of pricing error reduction (relative to the true solution)[12]. Therefore, the red area measures the pricing error that can be attributed to imposing certainty equivalence in the First-Order (CE) solution, while the blue area measures the error that can be attributed to its linearization. As the red area suggests, between 60% and 100% of the error stemming from the certainty equivalence solution for maturities below five years can be reduced by the risk adjustment of the First-Order solution. For a maturity of 1 year, for instance, 86% of the error in the First-Order (CE) solution can be attributed to the presence of certainty equivalence itself and hence reduced by the First-Order approximation. The remaining error resulting from the First-Order solution, as indicated by the blue area, is inevitable in linear models.

Our results shed light on the key source of the weakness of the linear approximation from perturbation in discrete-time models which results from its certainty equivalence property rather than linearization. Thus, we can conclude that once the vice of certainty equivalence is discarded, similar to the First-Order approximation in continuous time, one may stay with linear models and at the same time account for risk in a reasonable manner.

6 Conclusions

In this paper we use the fact that certainty equivalence (CE) breaks in continuous-time stochastic non-linear models when their rational expectation solution is approximated by first-order perturbations. To this end, we use a production economy with stochastic shocks to productivity, extended with habit formation and capital adjustment costs, which are known to generate substantial risk effects (see Jermann, 1998). We study the economic implications of breaking certainty equivalence in the linear approximation to the continuous-time model and compare them to the associated effects obtained using a CE solution, which is similar to that obtained using a first-order perturbation for discrete-time models. The reason that CE already breaks in first order is that the continuous-time formulation allows us to compute expectations before building the perturbation solution, which is not possible in discrete time. Moreover, the perturbation approximation is built around the variance of the shocks that drive the economy, and not around the standard deviation, as it is commonly done for discrete-time models. We shed light and illustrate the differences in the perturbation solution of equivalent models in continuous and discrete time and find substantial effects of risk.

To quantify the effects economically, we compute the asset pricing implications and

[12]The accompanying web appendix presents an alternative decomposition according to which we break the pricing mismatch into: (i) the error stemming from certainty equivalence in the non-linear world, which would result from a non-linear certainty equivalent solution; and (ii) the error stemming from linearization under certainty equivalence. We conclude that the errors induced by certainty equivalence and those by linearization are similar under both decompositions, which suggests that the entire error stemming from certainty equivalence is removed by the First-Order approximation.

pricing errors for the continuous-time model and show that the first-order already captures about 90 percent relative to the CE solution which, by construction, neglects the effects of risk. The correction in slopes from the second-order approximation in continuous time turns out to be small compared to the constant correction obtained in first order. Therefore, the first-order approximation turns out to be especially useful in this environment in which risk matters but nonlinearities are negligible.

We provide intuition why the first-order perturbation solution in continuous time accounts for prudence and hence is not certainty equivalent. In fact, the risk effects captured by continuous-time perturbations materialize with lower orders of approximation than those required by their (standard) discrete-time counterparts. Most prominent is that the continuous-time perturbation solution has a constant correction in the first-order approximation, which appears only in a second-order approximation in the discrete-time model. Similarly, the correction in slopes appears in second-order approximation in the continuous-time version, while it only appears in the third-order approximation in the discrete-time model.

We show that the (constant) risk adjustment in the first-order approximation does not only reduce pricing errors. First, we illustrate the effects for the resulting policy functions and compare them to the numerically more costly nonlinear approach. While the coefficients that are not associated with risk are close in discrete and continuous-time depending on the order of approximation, the risk corrections differ substantially. Second, we show how the risk adjustment affects the IRFs, which also reveals considerable differences mainly in the levels and for the computation of fixed points.

Our results encourage the use of continuous-time perturbations to account for risk in the class of (approximate) linear models. We believe this helps in the computation and estimation of large-scale macroeconomic models. Given the advantages of the continuous-time perturbation, future work should make these advantages more accessible by developing a toolbox that automates perturbation in continuous-time models.

Appendix

A Stochastic optimal control problem

A.1 The HJB equation and the first-order conditions

The benevolent planner chooses a path for consumption in order to maximize the expected discounted life-time utility of a representative household. Define the value of the optimal program as

$$V(K_0, X_0, A_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} U_0 \quad \text{s.t.} \quad (3) - (8)$$

in which $C_t \geq X_t \in \mathbb{R}^+$ denotes the control variable at instant $t \in \mathbb{R}^+$.

As a first step, we define the *Hamilton-Jacobi-Bellman equation* (HJB) for any $t \in [0, \infty)$

$$0 = \max_{C \geq X \in \mathbb{R}^+} \left\{ \frac{(C - X)^{1-\gamma}}{1 - \gamma} + \frac{1}{\mathrm{d}t} \mathbb{E}_t \mathrm{d}V(K, X, A) - \rho V(K, X, A) \right\}.$$

Itô's lemma imply

$$\mathrm{d}V(K, X, A) = V_K(K, X, A)\mathrm{d}K + V_X(K, X, A)\mathrm{d}X + V_A(K, X, A)\mathrm{d}A + \tfrac{1}{2} V_{AA}(K, X, A)\sigma_A^2 \mathrm{d}t$$

where $V_i(K, X, A) \equiv \frac{\partial V_i(K,X,A)}{\partial i}$, and $V_{ij}(K, X, A) \equiv \frac{\partial^2 V(K,X,A)}{\partial i \partial j}$ for $i, j = K, X, A$. Using the martingale difference properties of stochastic integrals, we arrive at

$$0 = \max_{C \geq X \in \mathbb{R}^+} \left\{ \frac{(C - X)^{1-\gamma}}{1 - \gamma} + \left(\Phi \left(\frac{\exp(A)K^\alpha - C}{K} \right) - \delta \right) K V_K(K, X, A) + (bC - aX) V_X(K, X, A) - \rho_A A V_A(K, X, A) + \tfrac{1}{2}\sigma_A^2 V_{AA}(K, X, A) - \rho V(K, X, A) \right\}.$$

The first-order condition for any interior solution reads

$$(C - X)^{-\gamma} + bV_X(K, X, A) = \Phi' \left(\frac{\exp(A)K^\alpha - C}{K} \right) V_K(K, X, A), \tag{55}$$

making optimal consumption an implicit function of the state variables, $C = C(K, X, A)$, where $\Phi'(\cdot) = a_1 (\cdot)^{-1/\xi}$. The maximized (concentrated) HJB equation is then

$$0 = \frac{(C(K,X,A) - X)^{1-\gamma}}{1-\gamma} + \left(\Phi\left(\frac{\exp(A)K^{\alpha} - C(K,X,A)}{K}\right) - \delta\right) K V_K(K,X,A)$$
$$+ (bC(K,X,A) - aX) V_X(K,X,A) - \rho_A A V_A(K,X,A)$$
$$+ \tfrac{1}{2}\sigma_A^2 V_{AA}(K,X,A) - \rho V(K,X,A). \quad (56)$$

A.2 Equilibrium in the time-space domain

Let $V_{ijl} \equiv \partial^3 V(K,X,A)/(\partial i \partial j \partial l)$ for any $i,j,l = K,X,A$. Then, using the maximized HJB equation in (56) together with the envelope theorem we obtain the associated costate variable with respect to capital, $V_{K,t}$,

$$\rho V_K = \Big(\Phi((\exp(A)K^{\alpha} - C)/K)K - \delta K\Big) V_{KK}$$
$$+ \Big(\Phi((\exp(A)K^{\alpha} - C)/K) + \Phi'((\exp(A)K^{\alpha} - C)/K)((\alpha-1)\exp(A)K^{\alpha-1}$$
$$+ C/K) - \delta\Big) V_K + \Big(bC - aX\Big) V_{XK} - \rho_A A V_{AK} + \tfrac{1}{2}\sigma_A^2 V_{AAK}. \quad (57)$$

On the other hand, the application of Itô's Lemma yields the evolution of (off-equilibrium) $V_{K,t}$ as

$$\mathrm{d}V_K = \bigg(\Big(\Phi\Big((\exp(A)K^{\alpha} - C)/K\Big) - \delta\Big) K V_{KK} + (bC - aX)V_{KX}$$
$$- \rho_A A V_{KA} + \tfrac{1}{2}\sigma_A^2 V_{KAA}\bigg)\mathrm{d}t + \sigma_A V_{KA}\mathrm{d}B_A. \quad (58)$$

Combining equations (57) and (58) we arrive at the following optimal/equilibrium stochastic differential equation (SDE) for V_K

$$\mathrm{d}V_K = \bigg(\rho - \Phi\Big((\exp(A)K^{\alpha} - C)/K\Big) - \Phi'\Big((\exp(A)K^{\alpha} - C)/K\Big)$$
$$\times \Big((\alpha-1)\exp(A)K^{\alpha-1} + C/K\Big) + \delta\bigg) V_K \mathrm{d}t + \sigma_A V_{KA}\mathrm{d}B_A. \quad (59)$$

Similarly, the optimal costate variable with respect to the habit level, $V_{X,t}$, reads

$$\rho V_X = -(C - X)^{-\gamma} + \Big(\Phi((\exp(A)K^{\alpha} - C)) - \delta\Big) K V_{KX,t}$$
$$+ (bC - aX)V_{XX} - aV_X - \rho_A A V_{AX} + \tfrac{1}{2}\sigma_A^2 V_{AAX}. \quad (60)$$

Using Itô's Lemma, the evolution of the (off-equilibrium) costate variable with respect to the habit level is given by

$$\mathrm{d}V_X = \Big(\Big(\Phi(\exp(A)K^\alpha - C/K) - \delta\Big)KV_{XK} + (bC - aX)V_{XX} \\ - \rho_A A V_{XA} + \tfrac{1}{2}\sigma_A^2 V_{XAA}\Big)\mathrm{d}t + \sigma_A V_{XA}\mathrm{d}B_A. \quad (61)$$

Combining equations (60) and (61) we arrive at the optimal/equilibrium SDE for V_X

$$\mathrm{d}V_X = \Big((\rho + a)V_X + (C - X)^{-\gamma}\Big)\mathrm{d}t + \sigma_A V_{XA}\mathrm{d}B_A. \quad (62)$$

Then, the equilibrium of the economy in the time-space domain can be characterized by the sequence $\{V_{K,t}, V_{X,t}, K_t, X_t, A_t\}_{t=0}^{\infty}$ that solves the following system of SDEs

$$\begin{aligned} \mathrm{d}V_{K,t} &= \Big(\rho - \Phi((\exp(A_t)K^\alpha - C_t)/K_t) - \Phi'((\exp(A_t)K_t^\alpha - C_t)/K_t) \\ &\qquad \times((\alpha - 1)\exp(A_t)K_t^{\alpha-1} + C_t/K_t) + \delta\Big)V_{K,t}\mathrm{d}t + \sigma_A V_{KA,t}\mathrm{d}B_{A,t} \\ \mathrm{d}V_{X,t} &= \Big((\rho + a)V_{X,t} + (Cv - X_t)^{-\gamma}\Big)\mathrm{d}t + \sigma_A V_{XA,t}\mathrm{d}B_{A,t} \\ \mathrm{d}K_t &= \Big(\Phi((\exp(A_t)K_t^\alpha - C_t)/K_t) - \delta\Big)K_t\mathrm{d}t \\ \mathrm{d}X_t &= (bC_t - aX_t)\mathrm{d}t \\ \mathrm{d}A_t &= -\rho_A A_t\mathrm{d}t + \sigma_A\mathrm{d}B_{A,t}, \end{aligned}$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$, and where C_t is the solution to the non-linear algebraic equation:

$$(C_t - X_t)^{-\gamma} + bV_{X,t} = \Phi'\left(\frac{\exp(A_t)K_t^\alpha - C_t}{K_t}\right)V_{K,t}.$$

A.3 Equilibrium in the state-space domain

Following Posch (2018), the equilibrium can be alternatively defined in the space of states by simply using the equilibrium partial differential equations (PDEs) for the costate variables in (57) and (60). Together with the first order condition in (55) they form a system of non-linear functional equations in the unknown policy functions $\{V_K, V_X, C\} = \{V_K(K, X, A), V_X(K, X, A), C(K, X, A)\}$

$$
\begin{aligned}
0 &= \left(\rho - \Phi\left(\frac{\exp(A)K^{\alpha} - C}{K}\right) - \Phi'\left(\frac{\exp(A)K^{\alpha} - C}{K}\right)\left(\frac{(\alpha-1)\exp(A)K^{\alpha-1} + C}{K}\right) + \delta\right)V_K \\
&\quad - \left(\Phi\left(\frac{\exp(A)K^{\alpha} - C}{K}\right) - \delta\right)KV_{KK} - \Big(bC - aX\Big)V_{XK} + \rho_A A V_{AK} - \tfrac{1}{2}\sigma_A^2 V_{AAK} \\
0 &= (\rho + a)V_X + (C - X)^{-\gamma} - \left(\Phi\left(\frac{\exp(A)K^{\alpha} - C}{K}\right) - \delta\right)KV_{KX} \\
&\quad -(bC - aX)V_{XX} + \rho_A A V_{AX} - \tfrac{1}{2}\sigma_A^2 V_{AAX} \\
0 &= (C - X)^{-\gamma} + bV_X - \Phi'\left(\frac{\exp(A)K^{\alpha} - C}{K}\right)V_K
\end{aligned}
$$

where the dynamics of the state variables are given by the controlled SDEs

$$
\begin{aligned}
\mathrm{d}K_t &= \Big(\Phi\big((\exp(A_t)K_t^{\alpha} - C_t\,(K_t, X_t, A_t))/K_t\big) - \delta\Big)\,K_t\mathrm{d}t \\
\mathrm{d}X_t &= \left(bC_t\,(K_t, X_t, A_t) - aX_t\right)\mathrm{d}t \\
\mathrm{d}A_t &= -\rho_A A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}
\end{aligned}
$$

subject to the initial conditions $K\,(0) = K_0$, $X\,(0) = X_0$ and $A\,(0) = A_0$.

A.4 Deterministic steady state

The deterministic steady state of the economy is given by the values $\{\overline{C}, \overline{I}, \overline{V}_K, \overline{V}_X, \overline{K}, \overline{X}, \overline{A}\}$ that solve the following system of equations

$$
\rho - \Phi(\overline{I}/\overline{K}) - \Phi'(\overline{I}/\overline{K})((\alpha-1)\exp(\overline{A})\overline{K}^{\alpha-1} + \overline{C}/\overline{K}) + \delta = 0 \tag{63}
$$

$$
(\rho + a)\overline{V}_X + (\overline{C} - \overline{X})^{-\gamma} = 0 \tag{64}
$$

$$
\Phi\left(\overline{I}/\overline{K}\right) - \delta = 0 \tag{65}
$$

$$
b\overline{C} - a\overline{X} = 0 \tag{66}
$$

$$
\left(\overline{C} - \overline{X}\right)^{-\gamma} + b\overline{V}_X - \Phi'\left(\overline{I}/\overline{K}\right)\overline{V}_K = 0 \tag{67}
$$

$$
\overline{I}/\overline{K} - \frac{\exp(\overline{A})\overline{K}^{\alpha} - \overline{C}}{\overline{K}} = 0 \tag{68}
$$

$$
\overline{A} = 0, \tag{69}
$$

which results from imposing $\sigma_A = 0$ together the idle condition $\mathrm{d}K_t/\mathrm{d}t = \mathrm{d}X_t/\mathrm{d}t = \mathrm{d}A_t/\mathrm{d}t = 0$ on the equilibrium PDEs (57) and (60).

The solution to this system of non-linear equations is entirely determined by the steady state value of the investment-capital ratio, $\overline{I}/\overline{K}$. Given the values of a_1 and a_2, it is possible to show that for any value of ξ

$$\overline{I}/\overline{K} = \delta.$$

Note that for the steady-state value of the investment-capital ratio, $\Phi(\delta) = \delta$, $\Phi'(\delta) = 1$, and $\Phi''(\overline{I}/\overline{K}) = \Phi''(\delta) = -1/(\xi\delta)$. From (63) and (68) we find the steady-state value of the capital stock as

$$\overline{K} = \left[\frac{\alpha \exp(\overline{A})}{(\rho+\delta)}\right]^{\frac{1}{1-\alpha}}. \tag{70}$$

Using (68) we find the steady-state value of consumption

$$\overline{C} = \exp(\overline{A})\overline{K}^{\alpha} - \delta\overline{K}. \tag{71}$$

From (66) we pin down the steady-state value of the habit as

$$\overline{X} = \frac{b}{a}\overline{C}. \tag{72}$$

Finally using (64) and (67) we find the steady-state values for the costate variables

$$\overline{V}_X = -\frac{1}{\rho+a}\left(\overline{C}-\overline{X}\right)^{-\gamma} \tag{73}$$

$$\overline{V}_K = \left(1-\frac{b}{\rho+a}\right)\left(\overline{C}-\overline{X}\right)^{-\gamma}. \tag{74}$$

A.5 Stochastic discount factor

When the habit is internal the agent takes into account the effect of today's consumption decisions on the future levels of habits. Following Detemple and Zapatero (1991),

$$m_t = ke^{-\rho t}\left\{ (C_t - X_t)^{-\gamma} - b\mathbb{E}_t\left[\int_t^{\infty} e^{-(\rho+a)(s-t)}\left(C_s - X_s\right)^{-\gamma} \mathrm{d}s\right]\right\}, \tag{75}$$

for some given constant k.

Using the (linear) equilibrium SDE for V_X in (62) we may write

$$e^{-(\rho+a)t}\mathrm{d}V_{X,t} = e^{-(\rho+a)t}\left((\rho+a)\,V_{X,t} + (C_t - X_t)^{-\gamma}\right)\mathrm{d}t + e^{-(\rho+a)t}V_{XA,t}\sigma_A \mathrm{d}B_{A,t},$$

or equivalently

$$e^{-(\rho+a)t}(\mathrm{d}V_{X,t} - (\rho+a)\,V_{X,t}\mathrm{d}t) = e^{-(\rho+a)t}\left(C_t - X_t\right)^{-\gamma}\mathrm{d}t + e^{-(\rho+a)t}V_{XA,t}\sigma_A \mathrm{d}B_{A,t}.$$

Note that Itô's formula yields

$$\mathrm{d}(e^{-(\rho+a)t}V_{X,t}) \;=\; -(\rho+a)e^{-(\rho+a)t}V_{X,t} + e^{-(\rho+a)t}\mathrm{d}V_{X,t}$$

such that

$$\mathrm{d}(e^{-(\rho+a)t}V_{X,t}) = e^{-(\rho+a)t}\left(C_t - X_t\right)^{-\gamma}\mathrm{d}t + e^{-(\rho+a)t}V_{XA,t}\sigma_A \mathrm{d}B_{A,t}.$$

Integrating both sides yields

$$\begin{aligned}
\int_t^T \mathrm{d}(e^{-(\rho+a)s}V_{X,s}) &= \int_t^T e^{-(\rho+a)s}\left(C_s - X_s\right)^{-\gamma}\mathrm{d}s + \int_t^T e^{-(\rho+a)s}V_{XA,s}\sigma_A \mathrm{d}B_{A,s} \\
\Leftrightarrow\; V_{X,t} &= e^{-(\rho+a)(T-t)}V_{X,T} - \int_t^T e^{-(\rho+a)(s-t)}\left(C_s - X_s\right)^{-\gamma}\mathrm{d}s \\
&\quad - \int_t^T e^{-(\rho+a)(s-t)}V_{XA,s}\sigma_A \mathrm{d}B_{A,s}.
\end{aligned}$$

Applying the expectation operator (assuming existence of the integrals) implies

$$\mathbb{E}_t\left[V_{X,t}\right] = e^{-(\rho+a)(T-t)}\mathbb{E}_t\left[V_{X,T}\right] - \mathbb{E}_t\left[\int_t^T e^{-(\rho+a)(s-t)}\left(C_s - X_s\right)^{-\gamma}\mathrm{d}s\right].$$

Further, by letting $\lim_{T\to\infty} e^{-(\rho+a)(T-t)}\mathbb{E}\left[V_{X,T}\right] = 0$, we may write

$$V_{X,t} \equiv \lim_{T\to\infty}\mathbb{E}_t\left[V_{X,t}\right] \;=\; -\mathbb{E}_t\left[\int_t^\infty e^{-(\rho+a)(s-t)}\left(C_s - X_s\right)^{-\gamma}\mathrm{d}s\right]$$

such that (75) can be written as

$$m_t = e^{-\rho t}\Big[\left(C_t - X_t\right)^{-\gamma} + bV_{X_t}\Big] \tag{76}$$

so that the SDF, as defined in Section 4, is

$$m_s/m_t = e^{-\rho(s-t)}\frac{\left(C_s - X_s\right)^{-\gamma} + bV_{X,s}}{\left(C_t - X_t\right)^{-\gamma} + bV_{X,t}}.$$

Using Itô's lemma, the dynamics of m_t is given by

$$\frac{\mathrm{d}m_t}{m_t} = \mu_{m,t}\mathrm{d}t + \sigma_{m,t}\mathrm{d}B_{A_t}, \tag{77}$$

where the drift and diffusion coefficients are

$$\mu_{m,t} = -\rho - \frac{(C_t - X_t)^{-\gamma}}{(C_t - X_t)^{-\gamma} + bV_{X,t}} \Big[\gamma (C_t - X_t)^{-1} \Big(\mu_{C,t} - (bC_t - aX_t) \Big) - b (C_t - X_t)^{\gamma} \mu_{V_X,t} - \tfrac{1}{2}\gamma(\gamma+1)(C_t - X_t)^{-2} \sigma_{C,t}^2 \Big] \quad (78)$$

and

$$\sigma_{m,t} = -\frac{(C_t - X_t)^{-\gamma}}{(C_t - X_t)^{-\gamma} + bV_{X,t}} \left[\gamma (C_t - X_t)^{-1} \sigma_{C,t} - b (C_t - X_t)^{\gamma} \sigma_{V_X,t} \right]. \quad (79)$$

Note that (78) and (79) depend on the drift and diffusion coefficients of the policy functions for consumption and the habit costate variable. Using Itô's lemma we can show that they are given by

$$\begin{aligned}
\mu_{C,t} &= \frac{\partial C_t}{\partial K_t} \left[\Phi \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) - \delta \right] K_t + \frac{\partial C_t}{\partial X_t} (bC_t - aX_t) - \frac{\partial C_t}{\partial A_t} \rho_A A_t + \tfrac{1}{2} \frac{\partial^2 C_t}{(\partial A_t)^2} \sigma_A^2 \\
\mu_{V_X,t} &= (\rho + a) V_{X,t} + (C_t - X_t)^{-\gamma} \\
\sigma_{C,t} &= \frac{\partial C_t}{\partial A_t} \sigma_A \\
\sigma_{V_X,t} &= \frac{\partial V_{X,t}}{\partial A_t} \sigma_A.
\end{aligned}$$

B The stochastic growth model

By letting $X_0 = b = 0$ and $\xi \to \infty$, the model in Section 2 collapses to

$$V(K_0, A_0) = \max_{\{C_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \mathrm{d}t \right]$$

subject to

$$\begin{aligned} \mathrm{d}K_t &= (\exp(A_t) K_t^{\alpha} - C_t - \delta K_t)\,\mathrm{d}t, \quad K_0 > 0 \\ \mathrm{d}A_t &= -\rho_A A_t \mathrm{d}t + \sqrt{\eta \sigma_A^2} \mathrm{d}B_{A,t}, \quad A_0 > 0, \end{aligned}$$

where we have included explicitly the perturbation parameter, η, that rescales the amount of variance in the model. The associated HJB equation reads

$$\begin{aligned} \rho V(K, A; \eta) = \max_{C} \Bigg\{ & \frac{C^{1-\gamma}}{1-\gamma} + (\exp(A) K^{\alpha} - C - \delta K) V_K(K, A; \eta) \\ & - \rho_A A V_A(K, A; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AA}(K, A; \eta) \Bigg\}. \end{aligned}$$

The first order condition for an interior solution is

$$C^{-\gamma} = V_K(K, A; \eta),$$

making optimal consumption a function of the state variables and the perturbation parameter, $C^{\star} = C(K, A; \eta)$. Substituting back, the maximized (concentrated) HJB equation reads

$$\begin{aligned} \rho V(K, A; \eta) = {} & \frac{C(K, A; \eta)^{1-\gamma}}{1-\gamma} + (\exp(A) K^{\alpha} - C(K, A; \eta) - \delta K) V_K(K, A; \eta) \\ & - \rho_A A V_A(K, A; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AA}(K, A; \eta) \end{aligned}$$

from which we can obtain the costate variable (using the envelope theorem) as

$$\begin{aligned} \rho V_K(K, A; \eta) = {} & (\exp(A) K^{\alpha} - C(K, A; \eta) - \delta K) V_{KK}(K, A; \eta) \\ & (\alpha \exp(A) K^{\alpha-1} - \delta) V_K(K, A; \eta) - \rho_A A V_{AK}(K, A; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AAK}(K, A; \eta) \end{aligned}$$

such that

$$\begin{aligned} (\rho - \alpha \exp(A) K^{\alpha-1} + \delta) V_K(K, A; \eta) = {} & (\exp(A) K^{\alpha} - C(K, A; \eta) - \delta K) V_{KK}(K, A; \eta) \\ & - \rho_A A V_{AK}(K, A; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AAK}(K, A; \eta). \end{aligned}$$

Using Itô's Lemma, the evolution of the costate variable is given by

$$\begin{aligned}
\mathrm{d}V_K(K,A;\eta) &= V_{KK}(K,A;\eta)\mathrm{d}K + V_{KA}(K,A;\eta)\mathrm{d}A + \tfrac{1}{2}\eta\sigma_A^2 V_{KAA}(K,A;\eta)\mathrm{d}t \\
&= (\rho - \alpha\exp(A)K^{\alpha-1} + \delta)V_K(K,A;\eta)\mathrm{d}t + V_{KA}(K,A;\eta)\sqrt{\eta\sigma_A^2}\mathrm{d}B_A.
\end{aligned}$$

Using once again the first-order condition, we may alternatively write

$$\mathrm{d}C_t^{-\gamma} = (\rho - \alpha\exp(A_t)K_t^{\alpha-1} + \delta)C_t^{-\gamma}\mathrm{d}t - \gamma C_t^{-\gamma-1}C_{A,t}\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t}$$

or

$$\frac{\mathrm{d}C_t}{C_t} = \left[\frac{1}{\gamma}\left(\alpha\exp(A_t)K_t^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\left(\frac{C_{A,t}}{C_t}\right)^2\eta\sigma_A^2\right]\mathrm{d}t + \left(\frac{C_{A,t}}{C_t}\right)\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t}, \tag{80}$$

which is the Euler equation for consumption in (37). Together with (34) and (35), they define the equilibrium of the economy in the time-space domain.

Alternatively, the equilibrium of this economy can be characterized in the space of states by eliminating time and stochastic shocks from the previous equilibrium system. To do so, note that Itô's lemma implies

$$\begin{aligned}
\mathrm{d}C_t &= C_{K,t}\mathrm{d}K_t + C_{A,t}\mathrm{d}A_t + \tfrac{1}{2}C_{AA,t}\eta\sigma_A^2\mathrm{d}t \\
&= C_{K,t}\left(\exp(A_t)K_t^{\alpha} - C_t - \delta K_t\right)\mathrm{d}t \\
&\quad - C_{A,t}\rho_A A_t\mathrm{d}t + C_{A,t}\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t} + \tfrac{1}{2}C_{AA,t}\eta\sigma_A^2\mathrm{d}t.
\end{aligned}\tag{81}$$

Combining (80) and (81) yields

$$\begin{aligned}
&\frac{1}{\gamma}\left(\alpha\exp(A_t)K_t^{\alpha-1} - \delta - \rho\right)C_t + \tfrac{1}{2}(1+\gamma)C_t\left(\frac{C_{A,t}}{C_t}\right)^2\eta\sigma_A^2 \\
&\qquad - C_{K,t}\left(\exp(A_t)K_t^{\alpha} - C_t - \delta K_t\right) + C_{A,t}\rho_A A_t - \tfrac{1}{2}C_{AA,t}\eta\sigma_A^2 = 0
\end{aligned}$$

which corresponds to the functional equation

$$\mathcal{H}\left(C, C_K, C_A, C_{AA}, K, A; \eta\right) = 0.$$

Since the policy function $C = C\left(K, A; \eta\right)$ is unknown, we approximate it by means of a k-th order perturbation around the deterministic steady state. Substituting into the functional $\mathcal{H}$ yield the new functional

$$F(K,A;\eta) = \mathcal{H}\left(C\left(K,A;\eta\right), C_K\left(K,A;\eta\right), C_A\left(K,A;\eta\right), C_{AA}\left(K,A;\eta\right), K, A; \eta\right) = 0.$$

Consider the case of $k = 1$. Hence, optimal consumption is approximated as

$$C(K, A; \eta) \approx \overline{C} + \overline{C}_K \left(K - \overline{K}\right) + \overline{C}_A \left(A - \overline{A}\right) + \overline{C}_\eta \eta,$$

where $\overline{C}$ is the deterministic steady-state value of consumption.

Let $\mathcal{H}_j$ denote the partial derivative of $\mathcal{H}(\cdot)$ with respect to its j-th element. Then, in order to find the yet unknown coefficients $\overline{C}_K$, $\overline{C}_A$ and $\overline{C}_\eta$ we compute

$$\begin{aligned}
F_K(K_t, A_t; \eta) &= \mathcal{H}_1 C_K + \mathcal{H}_2 C_{KK} + \mathcal{H}_3 C_{AK} + \mathcal{H}_4 C_{AAK} + \mathcal{H}_5 = 0 \\
F_A(K_t, A_t; \eta) &= \mathcal{H}_1 C_A + \mathcal{H}_2 C_{KA} + \mathcal{H}_3 C_{AA} + \mathcal{H}_4 C_{AAA} + \mathcal{H}_6 = 0 \\
F_\eta(K_t, A_t; \eta) &= \mathcal{H}_1 C_\eta + \mathcal{H}_2 C_{K\eta} + \mathcal{H}_3 C_{A\eta} + \mathcal{H}_4 C_{AA\eta} + \mathcal{H}_7 = 0,
\end{aligned}$$

which evaluated at the deterministic steady state reduces to a system of three non-linear equations in the three unknowns, $\overline{C_K}$, $\overline{C_A}$, $\overline{C_\eta}$, that can be solved recursively

$$\begin{aligned}
F_K\left(\overline{K}, \overline{A}; 0\right) &= \mathcal{H}_1 \overline{C}_K + \mathcal{H}_5 = 0 \\
F_A\left(\overline{K}, \overline{A}; 0\right) &= \mathcal{H}_1 \overline{C}_A + \mathcal{H}_6 = 0 \\
F_\eta\left(\overline{K}, \overline{A}; 0\right) &= \mathcal{H}_1 \overline{C}_\eta + \mathcal{H}_7 = 0,
\end{aligned}$$

where $\mathcal{H}_1$, $\mathcal{H}_5$, $\mathcal{H}_6$ and $\mathcal{H}_7$ are also functions of the unknowns. In particular,

$$F_K\left(\overline{K}, \overline{A}; 0\right) = \left(\alpha \exp(\overline{A}) \overline{K}^{\alpha-1} - \delta - \overline{C}_K\right) \frac{\overline{C}_K}{\overline{C}} - \frac{1}{\gamma} \alpha (\alpha - 1) \exp(\overline{A}) \overline{K}^{\alpha-2}.$$

Since this derivative must be zero, we arrive to the quadratic equation

$$\overline{C}_K^2 - \left(\alpha \exp(\overline{A}) \overline{K}^{\alpha-1} - \delta\right) \overline{C}_K + \frac{1}{\gamma} \alpha (\alpha - 1) \exp(\overline{A}) \overline{K}^{\alpha-2} \overline{C} = 0$$

with roots

$$\overline{C}_K = \frac{\left(\alpha \exp(\overline{A}) \overline{K}^{\alpha-1} - \delta\right)}{2} \pm \sqrt{\frac{\left(\alpha \exp(\overline{A}) \overline{K}^{\alpha-1} - \delta\right)^2 - 4 \frac{1}{\gamma} \alpha (\alpha - 1) \exp(\overline{A}) \overline{K}^{\alpha-2} \overline{C}}{4}}.$$

We pick the positive root since it is the only one that is consistent with a concave value function $V(K, A)$ in the capital stock. To see why, recall that the first-order condition

$$u'(C(K, A)) = V_K(K, A),$$

together with the assumptions on the utility function $u(C)$ imposes a necessary condition for concavity of the value function. A sufficient condition for concavity is given by the derivative of the first order condition

$$u''\left(C\left(K,A\right)\right)C_K\left(K,A\right)=V_{KK}\left(K,A\right)$$

which suggests that $V_{KK}\left(K,A\right)<0$ if and only if $C_K\left(K,A\right)>0$ given that $u''\left(C\right)<0$.

We now solve for $\overline{C}_A$ from

$$F_A\left(\overline{K},\overline{A};0\right)=\frac{\alpha\exp(\overline{A})\overline{K}^{\alpha-1}}{\gamma}\overline{C}-\overline{C}_K\left(\exp(\overline{A})\overline{K}^{\alpha}-\overline{C}_A\right)+\overline{C}_A\rho_A.$$

Since this derivative must be zero, we arrive at the following linear equation

$$\overline{C}_A=\frac{1}{\left(\overline{C}_K+\rho_A\right)}\left[\overline{C}_K\exp(\overline{A})\overline{K}^{\alpha}-\frac{\alpha\exp(\overline{A})\overline{K}^{\alpha-1}}{\gamma}\overline{C}\right]$$

which can be readily evaluated once $C_K\left(\bar{K},\bar{A};0\right)$ is computed from the first step.

Finally, we obtain $\overline{C}_\eta$ from

$$F_\eta\left(\overline{K},\overline{A};0\right)=\overline{C}_\eta\overline{C}_K+\tfrac{1}{2}\left(1+\gamma\right)\overline{C}\left(\frac{\overline{C}_A}{\overline{C}}\right)^2\sigma_A^2-\tfrac{1}{2}\overline{C}_{AA}\sigma_A^2,$$

and since $F_\eta\left(\bar{K},\bar{A};0\right)=0$, we arrive at the linear equation

$$\overline{C}_\eta=-\left(\overline{C}_K\right)^{-1}\left[\tfrac{1}{2}\left(1+\gamma\right)\overline{C}\left(\frac{\overline{C}_A}{\overline{C}}\right)^2-\tfrac{1}{2}\overline{C}_{AA}\right]\sigma_A^2.$$

As explained in the main text, in order to complete the computation of the first-order perturbation we need to compute the still unknown $\overline{C}_{AA}$. This is done by constructing the second order approximation to the deterministic version of the model which will result in a linear system of equations in $\overline{C}_{KK}$, $\overline{C}_{KA}$, $\overline{C}_{AK}$, $\overline{C}_{AA}$. Once solved, the first order approximation is complete.

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Risk Matters: Breaking Certainty Equivalence in Linear Approximations

Web Appendix

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September 2020

Contents

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C The discrete-time model

This appendix introduces an equivalent discrete-time version of the prototype RBC model studied in the paper. The model follows closely that in Jermann (1998). Table C1 gives a summary of the model setup in continuous and discrete time. We also provide a summary of the perturbation method for discrete-time economies in the spirit of Schmitt-Grohe and Uribe (2004); Fernández-Villaverde et al. (2016), and stress how certainty equivalence results from a first-order approximation. Finally, we discuss the concept of risky steady state and how to approximate it based on the work by de Groot (2013).

C.1 The social planner's problem

Consider the problem faced by a social planner with preferences over streams of consumption, C_t, which are summarized by the expected present discounted value of a representative agent's life time utility

$$\tilde{U}_0 \equiv \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \right], \tag{C.1}$$

where $\beta \in (0, 1)$ is the subjective discount factor. We further assume that consumption is a non-negative choice that cannot fall below a subsistence level, $C_t \geq X_t$, where X_t denotes internal habits in consumption. Following Grishchenko (2010), the household's internal habit is defined as

$$X_t = \tilde{b} \sum_{s=0}^{t-1} (1 - \tilde{a})^{t-s-1} C_s$$

or equivalently,

$$X_t = \tilde{b} C_{t-1} + (1 - \tilde{a}) X_{t-1}. \tag{C.2}$$

The parameters $\tilde{a}$ and $\tilde{b}$ share the same interpretation as in the main text, although a tilde on top of the parameters indicates that their value might not be the same due to the discrete-time nature of the problem. Note that once again the household preferences collapse to the standard time-separable case if $X_0 = \tilde{b} = 0$.

The aggregate output of the economy is produced using the Cobb-Douglas technology

$$Y_t = \exp(A_t) K_t^{\alpha} L_t^{1-\alpha}, \tag{C.3}$$

where K_t is the aggregate capital stock, and L_t is the perfectly inelastic labor supply (normalized to one $\forall t \geq 0$). The former accumulates according to

$$K_{t+1} = \Phi \left(\frac{I_t}{K_t} \right) K_t + (1 - \delta) K_t, \quad K_0 > 0, \tag{C.4}$$

where

$$\Phi(I_t/K_t) = \frac{a_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{1-1/\xi} + a_2, \tag{C.5}$$

represents adjustment costs of adjusting capital. On the other hand, total factor productivity (TFP), A_t, is assumed to follow the AR(1) process

$$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \quad A_0 > 0, \tag{C.6}$$

where $\tilde{\rho}_A \in (0,1)$ measures the degree of persistence of technology, $\tilde{\sigma}_A > 0$ its volatility, and $\epsilon_{A,t} \sim \mathcal{N}(0,1)$ is a productivity shock. Finally, the economy satisfies the aggregate resource constraint

$$Y_t = C_t + I_t. \tag{C.7}$$

The problem faced by the social planner is that of choosing the time path for consumption that maximizes (C.1) subject to the dynamic constraints (C.2),(C.4), and (C.6), and the static constraints (C.3), (C.5), and (C.7):

$$\widetilde{V}(K_0, A_0, X_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} \tilde{U}_0 \quad \text{s.t.} \quad (C.2) - (C.7), \tag{C.8}$$

in which $C_t \geq X_t \in \mathbb{R}^+$ denotes the control variable at time $t \in \mathbb{Z}$, and $\widetilde{V}_0 \equiv \widetilde{V}(K_0, X_0, A_0)$ the value of the optimal plan (value function) from the perspective of time $t = 0$. For any $t \in \{0, 1, 2, \dots\}$, a necessary condition for optimality is given by the *Bellman equation*

$$\widetilde{V}(K_t, A_t, X_t) = \max_{C_t \geq X_t \in \mathbb{R}^+} \left\{ \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \widetilde{V}(K_{t+1}, A_{t+1}, X_{t+1}) \right\} \tag{C.9}$$

subject to

$$\begin{aligned} K_{t+1} &= \Phi\left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t}\right) K_t + (1-\delta) K_t \\ X_{t+1} &= \tilde{b} C_t + (1-\tilde{a}) X_t \\ A_{t+1} &= \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1}. \end{aligned}$$

The first order condition for an interior solution is

$$(C_t - X_t)^{-\gamma} + \tilde{b}\beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1}\right] = \Phi'\left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t}\right) \beta \mathbb{E}_t \left[\widetilde{V}_{K,t+1}\right], \tag{C.10}$$

where $\widetilde{V}_{K,t+1} \equiv \widetilde{V}_K(K_{t+1}, X_{t+1}, A_{t+1})$, $\widetilde{V}_{X,t+1} \equiv \widetilde{V}_X(K_{t+1}, X_{t+1}, A_{t+1})$, and $\widetilde{V}_{A,t+1} \equiv \widetilde{V}_A(K_{t+1}, X_{t+1}, A_{t+1})$ are the partial derivatives of the value function with respect to each of the states. Equation (C.10) makes optimal consumption an implicit function of the state variables, $C_t^{\star} = C(K_t, A_t, X_t)$.

By means of the Envelope theorem, the costate variable with respect to capital is defined by

$$\widetilde{V}_{K,t} = \beta \left(\Phi' \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) \left((\alpha - 1) \exp(A_t) K_t^{\alpha - 1} + \frac{C_t}{K_t} \right) \right.$$
$$\left. + \Phi \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) + 1 - \delta \right) \mathbb{E}_t \left[\widetilde{V}_{K,t+1} \right],$$

while with respect to the habit by

$$\widetilde{V}_{X,t} = -\left(C_t - X_t\right)^{-\gamma} + (1 - \tilde{a}) \beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1} \right].$$

A solution to the planner's problem is given by the sequence $\left\{ \widetilde{V}_{K,t}, \widetilde{V}_{X,t}, K_t, X_t, A_t \right\}_{t=0}^{\infty}$ that solves the boundary value problem (with appropriate transversality conditions) characterized by the system of equilibrium stochastic difference equations:

$$\begin{aligned} \widetilde{V}_{K,t} &= \beta \left(\Phi' \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) \left((\alpha - 1) \exp(A_t) K_t^{\alpha - 1} + \frac{C_t}{K_t} \right) \right. \\ &\quad \left. + \Phi \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) + 1 - \delta \right) \mathbb{E}_t \left[\widetilde{V}_{K,t+1} \right] && \text{(C.11)} \\ \widetilde{V}_{X,t} &= -\left(C_t - X_t\right)^{-\gamma} + (1 - \tilde{a}) \beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1} \right] && \text{(C.12)} \\ X_{t+1} &= \tilde{b} C_t + (1 - \tilde{a}) X_t && \text{(C.13)} \\ K_{t+1} &= \Phi \left(\left(\exp(A_t) K_t^{\alpha} - C_t \right) / K_t \right) K_t + (1 - \delta) K_t && \text{(C.14)} \\ A_{t+1} &= \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1} && \text{(C.15)} \end{aligned}$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$, and where C_t solves the non-linear algebraic equation in (C.10).

Table C1 gives a summary of the model setup in continuous and discrete time.

C.2 Deterministic steady state

In the absence of uncertainty ($\tilde{\sigma}_A = 0$), the deterministic steady state is defined as an equilibrium in which all variables in the economy are constant. Hence, given the assumptions on the capital adjustment cost function in (C.5), the deterministic steady

	Continuous-time	Discrete-time
Objective function	$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$	$\mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\right]$
Market clearing	$\exp(A_t)K_t^\alpha L^{1-\alpha}=C_t+I_t$	$\exp(A_t)K_t^\alpha L^{1-\alpha}=C_t+I_t$
Capital dynamics	$\mathrm{d}K_t=\left(\Phi\left(\frac{I_t}{K_t}\right)-\delta\right)K_t\mathrm{d}t$	$K_{t+1}=\left(\Phi\left(\frac{I_t}{K_t}\right)+(1-\delta)\right)K_t$
Habit dynamics	$\mathrm{d}X_t=(bC_t-aX_t)\,\mathrm{d}t$	$X_{t+1}=\tilde{b}C_{t-1}+\tilde{a}X_{t-1}$
TFP dynamics	$\mathrm{d}A_t=-\rho_A A_t\mathrm{d}t+\sigma_A\mathrm{d}B_{A,t}$	$A_{t+1}=\tilde{\rho}_A A_t+\tilde{\sigma}_A\epsilon_{A,t+1}$
TFP shock	$(B_{A,t+\Delta}-B_{A,t})\sim N(0,\Delta)$	$\epsilon_{A,t}\sim N(0,1)$

Table C1. Summary of the two modeling frameworks. The table summarizes the two modeling frameworks in continuous and discrete time.

state is fully characterized by

$$\overline{A} = 0 \tag{C.16}$$

$$\overline{K} = \left[\frac{\alpha\exp(\overline{A})}{\rho+\delta}\right]^{\frac{1}{1-\alpha}} \tag{C.17}$$

$$\overline{C} = \exp(\overline{A})\overline{K}^\alpha-\delta\overline{K} \tag{C.18}$$

$$\overline{X} = \frac{b}{a}\overline{C} \tag{C.19}$$

$$\beta\overline{\overline{V}}_X = -\frac{1}{\rho+a}\left(\overline{C}-\overline{X}\right)^{-\gamma} \tag{C.20}$$

$$\beta\overline{\overline{V}}_K = \left(1-\frac{b}{\rho+a}\right)\left(\overline{C}-\overline{X}\right)^{-\gamma}, \tag{C.21}$$

where $\overline{\overline{V}}_X$ and $\overline{\overline{V}}_K$ denote the deterministic steady-state values of the costate variables for the capital stock and the habit formation in the discrete-time economy. By setting $\beta=1/(1+\rho)$, and $\tilde{b}=b$ and $\tilde{a}=a$, we ensure that the steady state values of the capital stock and the long-run habit-to-consumption ratio are equal in the discrete- and continuous-time models.

C.3 Perturbation method

The equilibrium conditions of the model are summarized by equations (C.11)–(C.15). As in the continuous-time case, the policy functions that solve these conditions are not available in closed form and therefore will be approximated using perturbation methods.

As before, let the augmented stochastic process for the TFP be given by

$$A_{t+1} = \tilde{\rho}_A A_t + \eta \tilde{\sigma}_A \epsilon_{A,t+1},$$

where η is the perturbation parameter that controls the standard deviation of TFP shocks (not the variance as in the continuous-time model).

Following Schmitt-Grohe and Uribe (2004), the equilibrium conditions can be compactly written as

$$\mathbb{E}_t \left[\mathcal{H}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t; \eta) \right] = \mathbf{0}, \tag{C.22}$$

where $\mathbf{x}_t = [K_t, X_t, A_t]^\intercal$ is the vector of state variables at time t, with initial value $\mathbf{x}_0 > \mathbf{0}$, $\mathbf{y}_t = \left[\widetilde{V}_{K,t}, \widetilde{V}_{X,t}, \widetilde{V}_{A,t}, C_t \right]^\intercal$ is the vector of control variables at time t, and $\mathcal{H}$ is an operator that collects the equilibrium conditions (C.11)–(C.15). The deterministic steady state is then defined as the pair $(\overline{\mathbf{y}}, \overline{\mathbf{x}})$ that solves

$$\mathcal{H}(\overline{\mathbf{y}}, \overline{\mathbf{y}}, \overline{\mathbf{x}}, \overline{\mathbf{x}}; 0) = \mathbf{0}. \tag{C.23}$$

The solution to the discrete-time model in (C.22) takes the form

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t; \eta) \tag{C.24}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}, \tag{C.25}$$

where $\mathbf{g}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{y}_t$, and $\mathbf{h}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{x}_{t+1}$. Substituting into the functional operator that defines the equilibrium delivers the new operator

$$F(\mathbf{x}_t; \eta) \equiv \mathbb{E}_t \left[\mathcal{H} \left(\mathbf{g}(\mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}; \eta), \mathbf{g}(\mathbf{x}_t; \eta), \mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}, \mathbf{x}_t; \eta \right) \right] = \mathbf{0}. \tag{C.26}$$

A perturbation-based approximation to the solution of problem (C.22) builds a Taylor series expansion of the unknown policy functions around the deterministic steady state using the fact that (C.26) holds for any values of $\mathbf{x}_t$ and η. An immediate consequence of the latter is that all the partial derivatives of the functional $F(\mathbf{x}_t; \eta)$ must be zero, i.e.,

$$F_{x_i^k \eta^j}(\mathbf{x}_t; \eta) = 0, \quad \forall x, \eta, i, k, j,$$

where $F_{x_i^k \eta^j}(\mathbf{x}_t; \eta)$ denotes the derivative of F with respect to the i-th element in $\mathbf{x}_t$ taken k times, and with respect to η taken j times evaluated at $(\mathbf{x}_t; \eta)$.

A first-order approximation to the policy functions is given by

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) &\approx \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_\eta(\overline{\mathbf{x}};0)\eta\\ \mathbf{h}(\mathbf{x}_t;\eta) &\approx \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_\eta(\overline{\mathbf{x}};0)\eta,\end{aligned}$$

where $\mathbf{g}(\overline{\mathbf{x}};0)$ and $\mathbf{h}(\overline{\mathbf{x}};0)$ correspond to the deterministic steady-state values of the control and state variables derived from (C.23), and where the constants $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_\eta(\overline{\mathbf{x}};0)$, $\mathbf{h}_\eta(\overline{\mathbf{x}};0)$ can be determined by solving the system of equations formed by

$$\begin{aligned}F_{x_i}(\overline{\mathbf{x}};0) &= 0, \quad \forall i\\ F_\eta(\overline{\mathbf{x}};0) &= 0.\end{aligned}$$

We refer to the first set of equations (those not involving the perturbation parameter) as the perfect-foresight component of the approximation, and to the second set of equations as the stochastic component of the approximation (cf. Andreasen and Kronborg, 2018).

The system of equations resulting from the perfect-foresight component is quadratic in the unknowns $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$. We pick the solution that ensures stability of the model's endogenous variables, i.e., the stable manifold (e.g., Blanchard and Kahn, 1980; Klein, 2000). The remaining constants, $\mathbf{g}_\eta(\overline{\mathbf{x}};0)$ and $\mathbf{h}_\eta(\overline{\mathbf{x}};0)$, correspond to the solution of the system of equations formed by the stochastic component, the unique solution being $\mathbf{g}_\eta(\overline{\mathbf{x}};0)=\mathbf{h}_\eta(\overline{\mathbf{x}};0)=0$ (cf. Fernández-Villaverde et al., 2016). Hence

$$\mathbf{g}(\mathbf{x}_t;\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\bar{\mathbf{x}}) \tag{C.27}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) \approx \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\bar{\mathbf{x}}), \tag{C.28}$$

implying that up to a first order, the approximation exhibits certainty equivalence, i.e., the solution of the model is identical to the solution of the same model in the absence of uncertainty, $\eta=0$.

Similarly, a second-order approximation to the policy functions is given by

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) \approx{}& \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_\eta(\overline{\mathbf{x}};0)\eta\\ &+\tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes\eta+\tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2\end{aligned}$$

and

$$\begin{aligned}\mathbf{h}(\mathbf{x}_t;\eta) \approx{}& \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_\eta(\overline{\mathbf{x}};0)\eta\\ &+\tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes\eta+\tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2,\end{aligned}$$

where the definition of the matrices $\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$, and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$ can be found in Binning (2013). These unknown coefficients correspond to the solution of the system of equations formed by

$$\begin{aligned} F_{x_i x_j}(\overline{\mathbf{x}};0) &= 0 \quad \forall i,j, \\ F_{\eta\eta}(\overline{\mathbf{x}};0) &= 0. \end{aligned}$$

As shown in Schmitt-Grohe and Uribe (2004), the cross derivatives $\mathbf{g}_{\mathbf{x}\eta}$ and $\mathbf{h}_{\mathbf{x}\eta}$ evaluated at $(\overline{\mathbf{x}};0)$ are zero, and hence the second-order perturbation reduces to

$$\mathbf{g}(\mathbf{x}_t;\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0) \quad \text{(C.29)}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) \approx \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0). \quad \text{(C.30)}$$

Hence, solving a second-order approximation introduces a constant correction in the policy functions that account for the effects of risk given by $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$, while the slopes of the policy functions are not affected by risk as $\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = \mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = 0$.

C.4 Calibration

For the numerical exercises presented in the paper we calibrate the discrete-time model as in the continuous-time case. In particular, we set the risk aversion parameter and the share of capital income to $\gamma = 2$ and $\alpha = 0.36$, respectively. The annual values for the subjective discount rate and the depreciation rate are fixed to $\beta = 1/(1+\rho) = 0.9606$ and $\delta = 0.0963$, respectively. For the habit process we use $a = 1$ and $b = 0.82$, while the adjustment cost parameter is calibrated to $\xi = 0.3261$. Finally, following Christensen et al. (2016), the annual values for the persistence and volatility of the TFP are set to $\tilde{\rho}_A = 0.8145$ and $\tilde{\sigma}_A = 0.0278$, respectively.

C.5 Risky steady state

Following de Groot (2013), it is possible to approximate the risky steady state of a discrete-time economy by making use of the second-order approximation around the deterministic steady state. First, consider the second-order approximation to the transition equation for the state variables in (C.30)

$$\mathbf{x}_{t+1} = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0) + \tilde{\sigma}_A \epsilon_{A,t+1}.$$

By setting the random disturbances to zero, $\epsilon_{A,t+1} = 0$, we compute the risky steady-state value of the state variables as the vector $\hat{\mathbf{x}}$ that satisfies $\mathbf{x}_{t+1} = \mathbf{x}_t = \hat{\mathbf{x}}$, and thus that solves the quadratic equation

$$\hat{\mathbf{x}} = \overline{\mathbf{x}} + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0).$$

Once $\hat{\mathbf{x}}$ is computed, it is possible to back out the implied risky steady-state value for the control variables, $\hat{\mathbf{y}}$, by simply inserting $\hat{\mathbf{x}}$ into (C.29)

$$\hat{\mathbf{y}} = \overline{\mathbf{y}} + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0).$$

The corresponding risky steady state values for habit, capital stock, and consumption resulting from the calibration in Section C.4 are $\hat{X} = 1.0608$, $\hat{K} = 4.7184$, and $\hat{C} = 1.2936$, respectively.

D Policy and Impulse-Response functions

For comparison purposes, this appendix reports the policy and impulse-response functions obtained from the discrete-time model in Appendix C. They are computed using the software platform `dynare`.

Figure D1 compares approximated policy functions for consumption across orders of approximation; on the left-hand side (LHS) for the continuous-time case and on the right-hand side (RHS) for the discrete-time case. Note that our calibration implies identical deterministic steady states across time assumptions. The policy function for consumption approximated by means of a first-order perturbation in the discrete-time model (solid line on the RHS) goes through the deterministic steady state (approximation point) which suggests that the approximation is certainty equivalent. In contrast, the First-Order approximation of the policy function in the continuous-time model (solid line on the LHS) does not go through the deterministic steady state (approximation point) indicating that it is not certainty equivalent. Only by shutting down the risk-correction, $\mathbf{g}_\eta(\overline{\mathbf{x}}; 0) = 0$, the continuous-time model First-Order CE approximation (dotted line on the LHS) will go through the deterministic steady state. Hence, as claimed in the main text, the First-Order CE resembles the first-order approximation in discrete time. Further note that in continuous time the First-Order delivers an approximation that is close to that provided by the Second-Order approximation (dashed line on the LHS) in the neighborhood of the deterministic steady state. The same does not occur in discrete time.

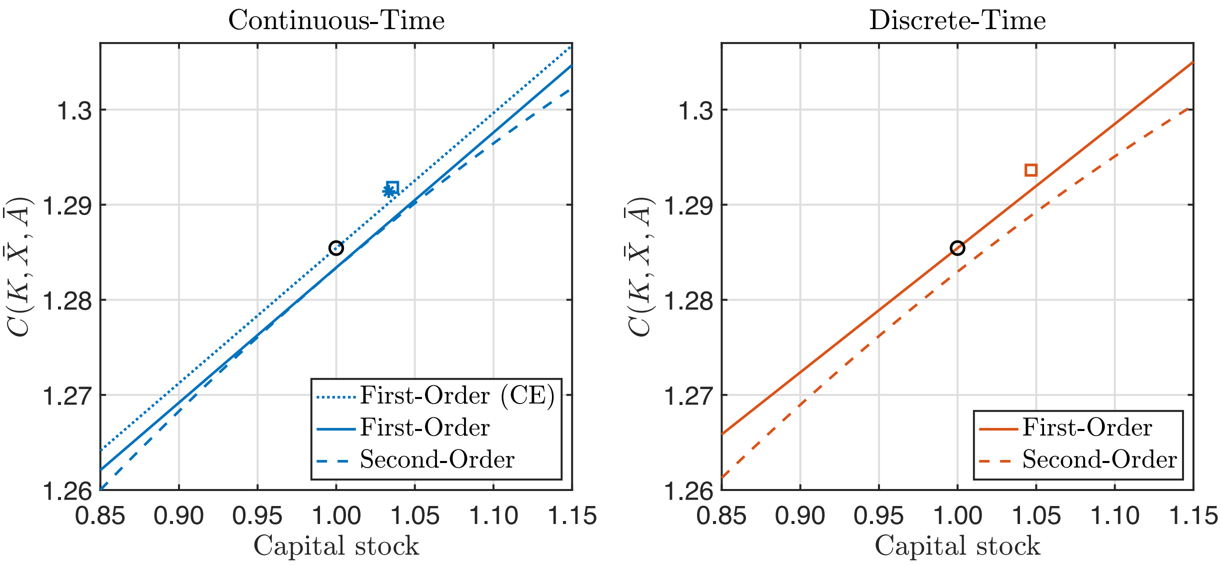

Figure D1. Continuous- and discrete-time approximated policy functions: First- and second-order approximations of the policy function for consumption around the deterministic steady state along the capital lattice while keeping habit and productivity at their deterministic steady-state values, $C\left(K, \overline{X}, \overline{A}\right)$. A circle denotes the deterministic steady state, a star denotes the first-order approximation and a square the second-order approximation of the risky steady state.

Figure D2 plots the approximated IRFs for consumption to a one standard deviation[1] shock in TFP across orders of approximation: on the LHS the continuous-time case and on the RHS the discrete-time case. As the first-order approximation in discrete time (solid line on the RHS) is certainty equivalent, the corresponding IRF starts in the deterministic steady state, where it also converges to. Comparing this IRF to the IRF from the First-Order CE in continuous time (dotted line on the LHS), one concludes that they are similar. Further, note that on the RHS we observe a large difference between first- and second-order approximated IRFs (solid vs. dashed line), since in the discrete time case only a second-order approximation provides risk-correction. In contrast, the differences between the IRFs resulting from the First- and Second-Order approximation (solid vs. dashed line on the LHS) are minor in the continuous time case, which reflects the fact that both approximations of the policy function are similar in the neighborhood of the deterministic steady state (see Figure D1). All these considerations suggest that the main weakness of the first-order approximation in discrete time is not that it is linear, but rather that it is certainty equivalent. Therefore, the continuous-time First-Order approximation is especially useful in situations in which risk matters but nonlinearities are negligible.

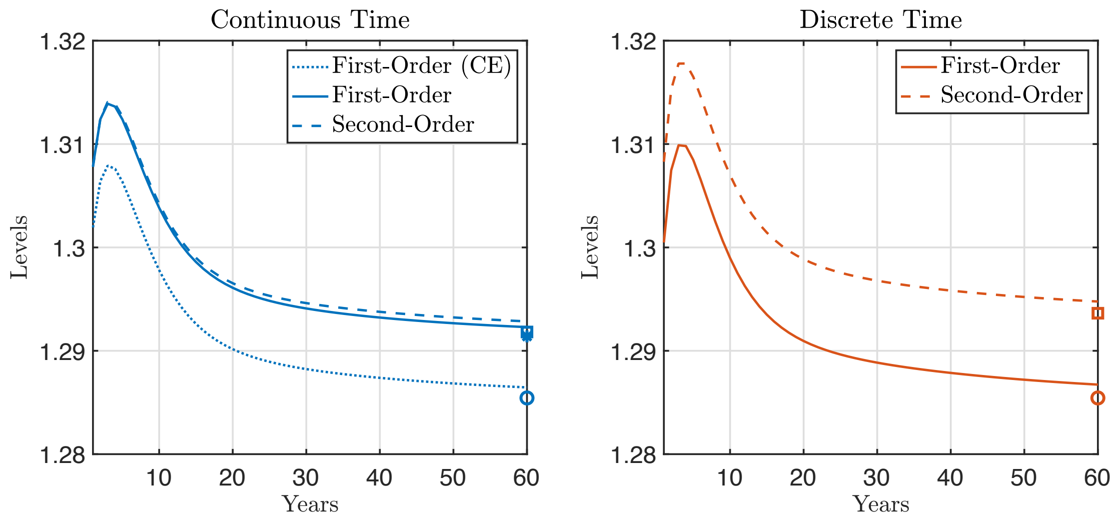

Figure D2. Impulse-Response function to a one s.d. shock in TFP (Discrete-time model): It plots the impulse response functions (IRFs) for the levels of aggregate consumption, capital, and habit when time is discrete. The variables in the economy are assumed to be in their corresponding risky steady states before the shock hits. A circle denotes the deterministic steady state, a star denotes the first-order approximation and a square the second-order approximation of the risky steady state.

[1] More precisely, for ease of comparison, we impose in both time assumptions an impulse of one standard deviation of the continuous-time model, i.e. $\sigma_A = 0.0307$.

E Pricing errors

Figure E1 reports the percentage (absolute) pricing errors for different approximations under the assumption that the true data generating process is given by the global approximation to the nonlinear stochastic model. The First-Order (CE), First-Order and Second-Order have been already introduced in Figure 3 in the main text.

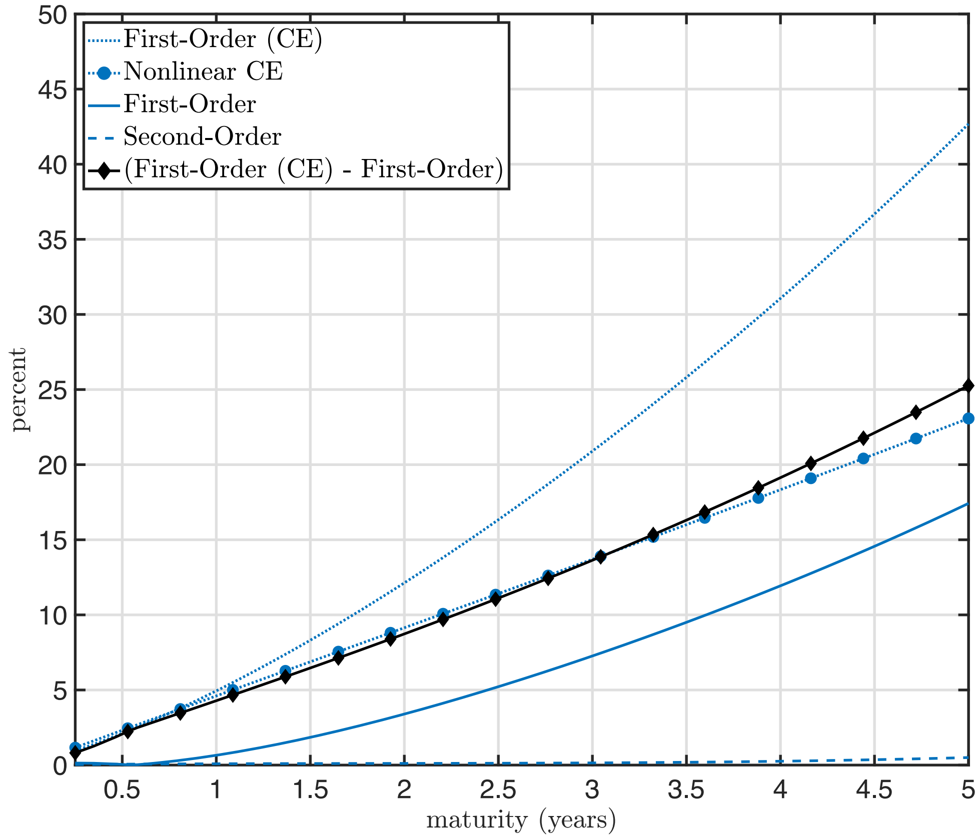

Figure E1. Decomposition of pricing errors: The graph plots the pricing errors resulting from First-Order (CE), First-Order, Nonlinear (CE), Second-Order, and the difference between the first two assuming that the true data generating process is the nonlinear stochastic solution.

Recall that the pricing error generated by the First-Order (CE) can be decomposed into: (i) the error stemming from the linearization of the nonlinear and stochastic policy function, which is captured by the First-Order approximation, and (ii) the error stemming from the imposition of certainty equivalence in the linear world. The latter is captured by the difference between First-Order (CE) and First-Order, and it is represented by the black line with diamonds: it measures the fraction of the pricing error that can be attributed to the imposition of certainty equivalence when using the First-Order (CE) solution. This measure can alternatively be interpreted as the reduction in the pricing errors that will be induced by the use of the (risk-sensitive) First-Order approximation.

Figure E1 presents an additional breakdown of the pricing errors generated by the use of the First-Order (CE). In particular, it is possible to decompose this error into: (i) the error stemming from imposing certainty equivalence in the nonlinear world, and

(ii) the error stemming from linearization in the presence of certainty equivalence. The former is given by the approximation of the policy function using a global method in a deterministic environment (Nonlinear CE, blue line with circles), while the latter would be given by the difference between the Nonlinear CE and the First-Order (CE).

By comparing the black line with diamonds and the blue line with circles, we can infer the effects from imposing certainty equivalence on the quality of the approximation. The first one provides a measure of this error in the linearized world, while the second one does it in the nonlinear world. The results suggest that the error reduction one would obtain from using the First-Order approximation is very close to the error one makes when imposing certainty equivalence in the nonlinear global solution. This can be interpreted as our First-Order approximation removing all of the error stemming from certainty equivalence such that all the remaining error can be attributed to linearization and, thus, is inevitable. Therefore, the First-Order approximation in continuous time makes it possible to account for the effects of risk in a linear framework.

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