Estimation of heterogeneous agent models: A likelihood approach*

Online Appendix (not for publication)

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Contents

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A Hamilton-Jacobi-Bellman equations

Define the optimal value function

$$V(a_0, e_0) = \max_{\{c_t\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (2.2), (2.3),$$

in which the general equilibrium factor rewards r and w are taken as parametric. Following the principle of optimality, the household's problem can be characterized by the Hamilton-Jacobi-Bellman equation

$$\rho V(a, e) = \max_{c \in \mathbb{R}^+} \left\{ u(c) + \frac{1}{\mathrm{d}t} \mathbb{E}_t \mathrm{d}V(a, e) \right\},$$

for any $t \in [0, \infty)$. Applying Itô's Lemma, or the change of variable formula (see Sennewald and Wälde, 2006), the continuation value is given by

$$\mathrm{d}V(a, e) = V_a(a, e)\,\mathrm{d}a + (V(a, e_l) - V(a, e_h))\mathrm{d}q_1 + (V(a, e_h) - V(a, e_l))\mathrm{d}q_2,$$

where $V_a(a, e)$ denotes the partial derivative of the value function with respect to wealth.

Using (2.2) together with the martingale difference properties of the stochastic integrals under Poisson uncertainty we have that for $s \leq t$

$$\begin{aligned} \mathbb{E}_s\Big[\int_s^t (V(a, e_l) - V(a, e_h))\mathrm{d}q_1 - \int_s^t (V(a, e_l) - V(a, e_h))\phi_1(e)\mathrm{d}t\Big] &= 0, \\ \mathbb{E}_s\Big[\int_s^t (V(a, e_h) - V(a, e_l))\mathrm{d}q_2 - \int_s^t (V(a, e_h) - V(a, e_l))\phi_2(e)\mathrm{d}t\Big] &= 0. \end{aligned}$$

Then, the Hamilton-Jacobi-Bellman equation can be written as

$$\begin{aligned} \rho V(a, e) = \max_{c \in \mathbb{R}^+} \Big\{ & u(c) + V_a(a, e)(ra + we - c) \\ & + (V(a, e_l) - V(a, e_h))\phi_1(e) + (V(a, e_h) - V(a, e_l))\phi_2(e) \Big\}. \end{aligned}$$

The first-order condition for an interior solution reads

$$u'(c) = V_a(a, e),$$

for any $t \in [0, \infty)$, making optimal consumption $c = c(a, e)$ a function only of the state variables and independent of calendar time, t.

Due to the state dependence of the arrival rates in the endowments of efficiency units, only one Poisson process will be active for each of the values of the discrete state variable, $e \in \mathcal{E}$. Using the first order condition we arrive to the bivariate system of maximized

HJB equations in Equations (2.7) and (2.8)

$$
\begin{aligned}
\rho V(a, e_h) &= u(c(a, e_h)) + V_a(a, e_h)(ra + we_h - c(a, e_h)) + (V(a, e_l) - V(a, e_h))\phi_{hl}, \\
\rho V(a, e_l) &= u(c(a, e_l)) + V_a(a, e_l)(ra + we_l - c(a, e_l)) + (V(a, e_h) - V(a, e_l))\phi_{lh}.
\end{aligned}
$$

B Fokker-Planck equations

Assume there exists a function f whose arguments are the stochastic processes a and e, and define the household's optimal savings function as $\dot{a} \equiv s(a,e) = ra + we - c(a,e)$. Using the change of variable formula, the evolution of f is given by

$$\mathrm{d}f(a,e) = f_a(a,e)\, s(a,e)\, \mathrm{d}t + (f(a,e_l) - f(a,e_h))\, \mathrm{d}q_1 + (f(a,e_h) - f(a,e_l))\, \mathrm{d}q_2.$$

Due to the state dependence of the arrival rates only one Poisson process will be active. Applying the expectations operator conditional on the information available at instant t and dividing by $\mathrm{d}t$ we obtain the infinitesimal generator of $f(a,e)$, denoted by $\mathbb{A}f(a,e) \equiv \mathbb{E}_t \mathrm{d}f(a,e)/\mathrm{d}t$, and given by

$$\frac{\mathbb{E}_t \mathrm{d}f(a,e)}{\mathrm{d}t} = f_a(a,e)\, s(a,e) + (f(a,e_l) - f(a,e_h))\, \phi_{hl} + (f(a,e_h) - f(a,e_l))\, \phi_{lh}. \tag{B.1}$$

By means of Dynkin's formula, the expected value of the function $f(\cdot)$ at a point in time t is given by the expected value of the function at instant $s < t$ plus the sum of the expected future changes up to t

$$\mathbb{E}f(a_t,e_t) = \mathbb{E}f(a_s,e_s) + \int_s^t \mathbb{E}(\mathbb{A}f(a_\tau,e_\tau))\, \mathrm{d}\tau. \tag{B.2}$$

Differentiating (B.2) with respect to time yields

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbb{E}f(a_t,e_t) &= \frac{\partial}{\partial t}\left\{\mathbb{E}f(a_s,e_s) + \int_s^t \mathbb{E}(\mathbb{A}f(a_\tau,e_\tau))\, \mathrm{d}\tau\right\} \\
&= \frac{\partial}{\partial t}\left\{\mathbb{E}f(a_s,e_s) + \int_s^t \mathbb{E}\mathrm{d}f(a_\tau,e_\tau)\right\} \\
&= \mathbb{E}(\mathbb{A}f(a,e)) \\
&= \sum_{e\in\{e_h,e_l\}} \int_{\underline{a}}^{\infty} \mathbb{A}f(a,e)\, g(a,e,t)\, \mathrm{d}a,
\end{aligned}$$

that is

$$\frac{\partial}{\partial t}\mathbb{E}f(a,e) = \underbrace{\int_{-\infty}^{\infty} \mathbb{A}f(a,e_h)\, g(a,e_h,t)\, \mathrm{d}a}_{\omega_{e_h}} + \underbrace{\int_{-\infty}^{\infty} \mathbb{A}f(a,e_l)\, g(a,e_l,t)\, \mathrm{d}a}_{\omega_{e_l}}, \tag{B.3}$$

where $g(a,e,t)$ is the joint density function of wealth and endowment of efficiency units at instant t. For illustration consider the case of $e = e_h$, i.e., ω_{e_h}. Using the definition of the infinitesimal operator together with (B.1) we note that

$$\mathbb{A}f(a, e_h) = f_a(a, e_h)\, s(a, e_h) + (f(a, e_l) - f(a, e_h))\, \phi_{hl}.$$

Hence,

$$\begin{aligned}
\omega_{e_h} &= \int_{\underline{a}}^{\infty} \Big[f_a(a, e_h)\, s(a, e_h) + (f(a, e_l) - f(a, e_h))\, \phi_{hl} \Big] g(a, e_h, t)\, \mathrm{d}a \\
&= \int_{\underline{a}}^{\infty} f_a(a, e_h)\, s(a, e_h)\, g(a, e_h, t)\, \mathrm{d}a + \int_{\underline{a}}^{\infty} (f(a, e_l) - f(a, e_h))\, \phi_{hl} g(a, e_h, t)\, \mathrm{d}a.
\end{aligned}$$

Using integration by part for the term associated with f_a

$$\int_{\underline{a}}^{\infty} f_a(a, e_h)\, s(a, e_h)\, g(a, e_h, t)\, \mathrm{d}a = -\int_{\underline{a}}^{\infty} f(a, e_h)\, \frac{\partial}{\partial a} [s(a, e_h)\, g(a, e_h, t)]\, \mathrm{d}a,$$

where

$$\frac{\partial}{\partial a} [s(a, e_h)\, g(a, e_h, t)] = \left(r - \frac{\partial}{\partial a} c(a, e_h) \right) g(a, e_h, t) + s(a, e_h)\, \frac{\partial}{\partial a} g(a, e_h, t).$$

It follows that

$$\begin{aligned}
\omega_{e_h} = \int_{\underline{a}}^{\infty} f(a, e_h) \left[-\left(r - \frac{\partial}{\partial a} c(a, e_h) \right) g(a, e_h, t) - s(a, e_h)\, \frac{\partial}{\partial a} g(a, e_h, t) \right] \mathrm{d}a \\
+ \int_{\underline{a}}^{\infty} \Big[(f(a, e_l) - f(a, e_h))\, \phi_{hl} \Big] g(a, e_h, t)\, \mathrm{d}a,
\end{aligned}$$

and

$$\begin{aligned}
\omega_{e_l} = \int_{\underline{a}}^{\infty} f(a, e_l) \left[-\left(r - \frac{\partial}{\partial a} c(a, e_l) \right) g(a, e_l, t) - s(a, e_l)\, \frac{\partial}{\partial a} g(a, e_2, t) \right] \mathrm{d}a \\
+ \int_{\underline{a}}^{\infty} \Big[(f(a, e_h) - f(a, e_l))\, \phi_{lh} \Big] g(a, e_l, t)\, \mathrm{d}a.
\end{aligned}$$

Note that the expected value of f can be written as

$$\mathbb{E}f(a,e) = \int_{\underline{a}}^{\infty} f(a,e_h)\, g(a,e_h,t)\, \mathrm{d}a + \int_{\underline{a}}^{\infty} f(a,e_l)\, g(a,e_l,t)\, \mathrm{d}a$$

and therefore

$$\frac{\partial}{\partial t}\mathbb{E}f(a,e) = \int_{\underline{a}}^{\infty} f(a,e_h)\, \frac{\partial}{\partial t} g(a,e_h,t)\, \mathrm{d}a + \int_{\underline{a}}^{\infty} f(a,e_l)\, \frac{\partial}{\partial t} g(a,e_l,t)\, \mathrm{d}a. \tag{B.4}$$

Finally we equate (B.3) and (B.4) and collect terms to obtain

$$\int_{\underline{a}}^{\infty} f(a,e_h)\, \varphi_{e_h} \mathrm{d}a + \int_{\underline{a}}^{\infty} f(a,e_l)\, \varphi_{e_l} \mathrm{d}a = 0, \tag{B.5}$$

where

$$\begin{aligned} \varphi_{e_h} = &- \left(r - \frac{\partial}{\partial a} c(a,e_h) + \phi_{hl} \right) g(a,e_h,t) \\ &\qquad - s(a,e_h) \frac{\partial}{\partial a} g(a,e_h,t) + \phi_{lh} g(a,e_l,t) - \frac{\partial}{\partial t} g(a,e_h,t), \end{aligned}$$

and

$$\begin{aligned} \varphi_{e_l} = &- \left(r - \frac{\partial}{\partial a} c(a,e_l) + \phi_{lh} \right) g(a,e_l,t) \\ &\qquad - s(a,e_l) \frac{\partial}{\partial a} g(a,e_l,t) + \phi_{hl} g(a,e_h,t) - \frac{\partial}{\partial t} g(a,e_l,t). \end{aligned}$$

The Fokker-Planck equations that define these subdensities are obtained by setting $\varphi_{e_l} = \varphi_{e_h} = 0$ since that is that only way the integral equation (B.5) can be satisfied for all possible functions f. A formal proof can be found in Bayer and Wälde (2010a,b). This results in a system of two non-autonomous quasi-linear partial differential equations in two unknowns $g(a,e_h,t)$, $g(a,e_l,t)$

$$\begin{aligned} \frac{\partial}{\partial t} g(a,e_h,t) + s(a,e_h) \frac{\partial}{\partial a} g(a,e_h,t) = & \\ &- \left(r - \frac{\partial}{\partial a} c(a,e_h) + \phi_{hl} \right) g(a,e_h,t) + \phi_{lh} g(a,e_l,t), \end{aligned}$$

$$\frac{\partial}{\partial t} g\left(a, e_l, t\right) + s\left(a, e_l\right) \frac{\partial}{\partial a} g\left(a, e_l, t\right) =$$
$$- \left(r - \frac{\partial}{\partial a} c\left(a, e_l\right) + \phi_{lh}\right) g\left(a, e_l, t\right) + \phi_{hl} g\left(a, e_h, t\right).$$

The stationary subdensities result when the time derivatives $\partial g\left(a, e, t\right)/\partial t$ are zero for all $e \in \mathcal{E}$, which transforms the previous system of equations into one of ordinary differential equations as described by (2.14) and (2.15).

Given the stationary subdensity function, the stationary probability "subdistributions" can be computed as

$$G\left(a, e\right) = \int_{\underline{a}}^{a} g\left(x, e\right) \mathrm{d}x, \tag{B.6}$$

where $G\left(a, e\right)$ denotes the probability that an individual with endowment of efficiency equal to $e \in \mathcal{E}$ has a wealth level of at most a. When $a \to \infty$, (2.11) implies that $\lim_{a \to \infty} G\left(a, e\right) = p\left(e\right)$. Similar to (2.12), the (unconditional) stationary probability distribution of wealth can be computed as

$$G\left(a\right) = G\left(a, e_h\right) + G\left(a, e_l\right), \tag{B.7}$$

which can be then used to compute the Gini coefficient in the economy

$$\mathcal{G} = \frac{1}{\mu} \int_{\underline{a}}^{\infty} G\left(a\right)\left(1 - G\left(a\right)\right) \mathrm{d}a, \tag{B.8}$$

where $\mu = \mathbb{E}\left(a\right)$ denotes the unconditional mean of wealth.

C Transition probabilities and the limiting distribution of income

Section C.1 of this appendix shows how to compute the transition probabilities at a given point in time across states using the arrival rates for the idiosyncratic income process. Section C.2 uses the two state income process of Section 2 to illustrate how to compute the limiting (stationary) probability distribution for the endowment of efficiency units defined in (2.13) from the arrival rates of the stochastic process defined by (2.3). A more detailed description can be found in Ross (2009).

C.1 Transition probabilities

In what follows, assume that the endowment of efficiency units (income) can take d different values. Let $p(e_i, e_j, t) \equiv \mathbb{P}(e_{t+s} = e_j \mid e_s = e_i)$ for all $s \leq t$ denotes the probability that the income process currently in state i will transit to state j at an instant later for all $i, j = 1, \ldots, d$. Let $\mathbf{P}(t) = [p(e_i, e_j, t)]_{1 \leq i,j \leq d}$ denote the corresponding stochastic transition probability matrix. Then, it is possible to show that the transition probabilities of a continuous time Markov Chain satisfy the system of Backward Kolmogorov equations

$$\dot{p}(e_i, e_j, t) = \sum_{k \neq i} \phi_{ik} p(e_k, e_j, t) - \nu_i p(e_i, e_j, t), \tag{C.9}$$

with $p(e_i, e_i, 0) = 1$ and $p(e_i, e_j, 0) = 0$ as initial conditions, and where $\dot{p}(e_i, e_j, t) = \lim_{s \to 0} \frac{1}{s} [p(e_i, e_j, t+s) - p(e_i, e_j, t)]$ for all i, j in the state space of efficiency units $\mathcal{E}$. Furthermore, $\phi_{ij} \geq 0$ denotes the instantaneous transition rates at which the labor efficiency process jumps from state j to state i, and $\nu_i = \sum_{j, j \neq i} \phi_{ij}$.

Associated with the intensity rates, let us define $\mathbf{\Phi}$ to be the generator, or transition intensity matrix with elements

$$\mathbf{\Phi} = \begin{cases} -\nu_i, & \text{if } i = j \\ \phi_{ij}, & \text{if } i \neq j. \end{cases}$$

Then, the system of Backward Kolmogorov equations in (C.9) has the following matrix representation

$$\dot{\mathbf{P}}(t) = \mathbf{\Phi} \mathbf{P}(t), \quad \mathbf{P}(0) = \mathbf{I},$$

with solution

$$\mathbf{P}(t) = \mathbf{P}(0) \exp(\mathbf{\Phi} t).$$

C.2 Stationary distribution

For illustration purposes suppose that the labor efficiency can take only two values, $e \in \{e_l, e_h\}$. Now, let us consider the case of an individual who is in state e_l at time s. Then, $p(e_l, e_h, t)$ denotes the probability that the individual's efficiency jumps from state e_l at time s to state e_h at time t. The instantaneous transition rates at which the stochastic process jumps to state e_l from state e_h, and to state e_h from state e_l, are given by ϕ_{hl} and ϕ_{lh}, respectively. Then, the transition probabilities at time t can be computed from the solution to the system of Backward Kolmogorov equations in (C.9)

$$\begin{aligned} \dot{p}(e_h, e_h, t) &= \phi_{hl}\left[p(e_l, e_h, t) - p(e_h, e_h, t)\right], \\ \dot{p}(e_l, e_h, t) &= \phi_{lh}\left[p(e_h, e_h, t) - p(e_l, e_h, t)\right] \end{aligned}$$

with initial conditions $p(e_h, e_h, s) = 1$ and $p(e_l, e_h, s) = 0$. The solution to this system of ordinary differential equations is given by

$$p(e_h, e_h, t) = \frac{\phi_{lh}}{\phi_{hl} + \phi_{lh}} + \frac{\phi_{hl}}{\phi_{hl} + \phi_{lh}} e^{-(\phi_{hl} + \phi_{lh})(t-s)} \tag{C.10}$$

$$p(e_l, e_h, t) = \frac{\phi_{lh}}{\phi_{hl} + \phi_{lh}} - \frac{\phi_{lh}}{\phi_{hl} + \phi_{lh}} e^{-(\phi_{hl} + \phi_{lh})(t-s)}. \tag{C.11}$$

Now let $p(e_h, s)$ denote the unconditional probability of being in state e_h at time s. The unconditional probability of being in the same state at time $t > s$ can be computed according to:

$$p(e_h, t) = p(e_h, s)\, p(e_h, e_h, t) + (1 - p(e_h, s))\, p(e_l, e_h, t). \tag{C.12}$$

In the limit as $t \to \infty$ the unconditional probability of having an endowment of high efficiency is given by:

$$\lim_{t \to \infty} p(e_h, t) = p(e_h) = \frac{\phi_{lh}}{\phi_{hl} + \phi_{lh}}. \tag{C.13}$$

A similar procedure can be used to show that the stationary and unconditional probability of having an endowment of low efficiency is:

$$\lim_{t \to \infty} p(e_l, t) = p(e_l) = \frac{\phi_{hl}}{\phi_{hl} + \phi_{lh}}. \tag{C.14}$$

The system of equations formed by (C.10) and (C.11) together with an appropriate choice of $(t - s)$ can be used to back out the instantaneous transition rates of the Poisson processes, ϕ_{hl} and ϕ_{lh} from any probability transition matrix. Given the annual frequency used in the calibration of the model of Section 2, we set $(t - s) = 1$ (one year).

D Computation of the stationary equilibrium

The computation of the stationary density of wealth is done following the method proposed in Achdou et al. (2020) which consists of two main blocks. The first block computes the stationary general equilibrium at the macro level by using the following fixed point algorithm in the time-invariant aggregate capital stock:

Algorithm D.1 (Stationary General Equilibrium) *Make an initial guess for the interest rate, $r^{(0)}$, and then for $j = 0, 1, \ldots$:*

1. *Compute the optimal consumption functions $c^{(j)}(a, e_h)$ and $c^{(j)}(a, e_l)$ and the subdensities $g^{(j)}(a, e_h)$ and $g^{(j)}(a, e_l)$.*

2. *Compute capital demand K^d and capital supply K^s.*

3. *Update $r^{(j+1)}$ using a combination of bisection, secant, and inverse quadratic interpolation methods.*

4. *If $\left\|K^s - K^d\right\| < \epsilon$ stop, otherwise return to step 1.*

Algorithm D.1 does not require to update the aggregate labor supply L at each iteration $j = 0, 1, \ldots$ since in our prototype economy the labor supply is assumed to be exogenous.

The second block approximates both the solution to the household's problem at the micro level and to the Fokker-Planck equations using the finite difference methods suggested in Candler (1999) and Achdou et al. (2020). These solutions, which are required in step 2 of Algorithm D.1 for every iteration $j = 0, 1, \ldots$, are computed in two independent stages. The first stages approximates the policy functions for consumption that solve the HJB equations (2.7) and (2.8), while the second stage approximates the subdensities of wealth that solve the Fokker-Planck equations (2.14) and (2.15).

D.1 Solving the Hamilton-Jacobi-Bellman equations.

Consider first the solution to the HJB equations. For each $e_t \in \mathcal{E}$, the finite difference method approximates the function $V(a_t, e_t)$ on an equally spaced grid for wealth with I discrete points, $a_i, i = 1, \ldots, I$, where $a_i \in \mathcal{A} = [a_{min}, a_{max}]$ and $a_{min} = \underline{a}$. The distance between points is denoted by Δa and we introduce the short-hand notation $V_{e,i} \equiv V(a_i, e)$. The derivative $V_a(a_i, e) \equiv V'_{e,i}$ is computed with either a forward or a backward difference approximation:

$$V'^{F}_{e,i} \approx \frac{V_{e,i+1} - V_{e,i}}{\Delta a} \tag{D.15}$$

$$V'^{B}_{e,i} \approx \frac{V_{e,i} - V_{e,i-1}}{\Delta a}. \tag{D.16}$$

Following Candler (1999), the choice of difference operator is based on an upwind differentiation scheme. The correct approximation is based on the direction of the continuous state variable. Thus, if the saving function, $s(a_i, e) \equiv s_{e,i} = ra_i + we - (u')^{-1}(V'_{e,i})$, is positive we use a forward operator and if it is negative we use the backward operator. This gives rise to the following upwind operator:

$$V'_{e,i} = V'^F_{e,i}\mathbf{1}_{\{s^F_{e,i}>0\}} + V'^B_{e,i}\mathbf{1}_{\{s^B_{e,i}<0\}} + \bar{V}'_{e,i}\mathbf{1}_{\{s^F_{e,i}<0<s^B_{e,i}\}} \tag{D.17}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function and, $s^F_{e,i}$ and $s^B_{e,i}$ the saving functions computed with the forward and difference operators respectively. Following Achdou et al. (2020), the concavity of the value function in the wealth dimension motivates the last term in (D.17) since there could be grid points $a_i \in \mathcal{A}$ for which $s^F_{e,i} < 0 < s^B_{e,i}$. In those cases, they suggest to set savings to be equal to zero which implies that the derivative of the value function is equal to $\bar{V}'_{e,i} = u'(ra_i + we)$.

The finite difference approximation to the HJB equations is then given by:

$$\rho V_{e,i} = u(c_{e,i}) + V'_{e,i}[ra_i + ew - c_{e,i}] + \phi_{-ee}[V_{-e,i} - V_{e,i}]$$

for each $e \in \mathcal{E}$, where optimal consumption is given by:

$$c_{e,i} = (u')^{-1}(V'_{e,i}),$$

and where the state-constraint boundary condition in (2.9) is enforced at the lower bound of the state space, a_{min}, by imposing $V'^B_{e,1} = u'(ra_1 + we)$.

The upwind representation of the HJB equation reads:

$$\rho V_{e,i} = u(c_{e,i}) + \frac{V_{e,i+1} - V_{e,i}}{\Delta a}(s_{e,i})^+ + \frac{V_{e,i} - V_{e,i-1}}{\Delta a}(s_{e,i})^- + \phi_{-ee}[V_{-e,i} - V_{e,i}] \tag{D.18}$$

where:

$$(s_{e,i})^+ = \max\left\{ra_i + we - (u')^{-1}\left(V'^F_{e,i}\right), 0\right\} \quad \text{and} \quad (s_{e,i})^- = \min\left\{ra_i + we - (u')^{-1}\left(V'^B_{e,i}\right), 0\right\}$$

denote the positive and negative parts of savings, respectively.

Equation (D.18) defines a highly non linear system of equations in $V_{e,i}$ that can only be solved by iterative methods. We follow Candler (1999) and set up an iterative procedure based on the time-dependent HJB equation, $V^l_{e,i} \equiv V(a_i, e, t)$. Then, from an arbitrary initial condition we integrate forward in time until the solution is no longer a function of the initial condition, i.e. until it converges to the time-independent HJB, $V_{e,i}$. The time-updating is carried out by means of an implicit scheme in which the value function at the next time step, $V_{e,i}^{l+1}$, is implicitly defined by the equation:

$$\frac{V_{e,i}^{l+1} - V_{e,i}^{l}}{\Delta} + \rho V_{e,i}^{l+1} = u\left(c_{e,i}^{l}\right) + \frac{V_{e,i+1}^{l+1} - V_{e,i}^{l+1}}{\Delta a}\left(s_{e,i}^{l}\right)^{+} + \frac{V_{e,i}^{l+1} - V_{e,i-1}^{l+1}}{\Delta a}\left(s_{e,i}^{l}\right)^{-} + \phi_{-ee}\left[V_{-e,i}^{l+1} - V_{e,i}^{l+1}\right] \quad \text{(D.19)}$$

where Δ is the time step size, $c_{e,i}^{l} = (u')^{-1}\left[\left(V_{e,i}^{l}\right)'\right]$, and $\left(V_{e,i}^{l}\right)'$ is given by (D.17).

Equation (D.19) constitutes a system of $2 \times I$ linear equations in $V_{e,i}^{l+1}$ with the following matrix representation:

$$\mathbf{A}^{l}\mathbf{V}^{l+1} = \mathbf{b}^{l} \quad \text{(D.20)}$$

where $\mathbf{V}^{l+1} = \left(V_{e_l,1}^{l+1}, \ldots, V_{e_l,I}^{l+1}, V_{e_h,1}^{l+1}, \ldots, V_{e_h,I}^{l+1}\right)'$, $\mathbf{b}^{l}$ is a vector with elements $b_{e,i}^{l} = u\left(c_{e,i}^{l}\right) + V_{e,i}^{l}/\Delta$ and $\mathbf{A}^{l}$ is the block matrix:

$$\mathbf{A}^{l} = \begin{bmatrix} \mathbf{A}_{e_l} & -\Phi_{hl} \\ -\Phi_{lh} & \mathbf{A}_{e_h} \end{bmatrix}$$

with $\Phi_{-ee} = -\phi_{-ee}\mathbf{I}_I$ and

$$\mathbf{A}_{e} = \begin{bmatrix} y_{e,1} & z_{e,1} & 0 & \cdots & 0 & 0 \\ x_{e,2} & y_{e,2} & z_{e,2} & \cdots & 0 & 0 \\ 0 & x_{e,3} & y_{e,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & y_{e,I-1} & z_{e,I-1} \\ 0 & 0 & 0 & \cdots & x_{e,I} & y_{e,I} \end{bmatrix}.$$

where

$$x_{e,i} = \frac{\left(s_{e,i}^{l}\right)^{-}}{\Delta a}, \quad y_{e,i} = \frac{1}{\Delta} + \rho + \frac{\left(s_{e,i}^{l}\right)^{+}}{\Delta a} - \frac{\left(s_{e,i}^{l}\right)^{-}}{\Delta a} + \phi_{-ee}, \quad z_{e,i} = -\frac{\left(s_{e,i}^{l}\right)^{+}}{\Delta a}.$$

and $e \in \mathcal{E}$. The iterative algorithm used to find the solution to the HJB equation can be summarized as follows:

Algorithm D.2 (Solution of the HJB equation) *Guess $V_{e,i}^{0}$ for each $e \in \mathcal{E}$ and $i = 1, \ldots, I$. Then for $l = 0, 1, 2, \ldots$:*

1. *Compute $\left(V_{e,i}^{l}\right)'$ using (D.17).*
2. *Compute $c_{e,i}^{l} = (u')^{-1}\left(V_{e,i}'\right)$.*
3. *Find $V_{e,i}^{l+1}$ by solving the system of equations defined in (D.20).*
4. *If $\left\|V_{e,i}^{l+1} - V_{e,i}^{l}\right\| < \epsilon$ stop. Otherwise, go to step 1.*

D.2 Solving the Fokker-Planck equations.

Once the optimal consumption has been computed from Algorithm D.2, we proceed to approximate the solution to the associated Fokker-Planck equations (2.14) and (2.15). As before, we use a finite difference method and apply it to:

$$0 = -\frac{\partial}{\partial a_t}\left[s\left(a_t, e_l\right) g\left(a_t, e_l\right)\right] - \phi_{hl} g\left(a_t, e_l\right) - \phi_{lh} g\left(a_t, e_h\right), \tag{D.21}$$

$$0 = -\frac{\partial}{\partial a_t}\left[s\left(a_t, e_h\right) g\left(a_t, e_h\right)\right] - \phi_{lh} g\left(a_t, e_h\right) - \phi_{hl} g\left(a_t, e_l\right) \tag{D.22}$$

which corresponds, as shown in Appendix B above, to an alternative representation of (2.14) and (2.15). We further need to restrict the solution to satisfy the integrability condition:

$$1 = \sum_{e_t \in \{e_l, e_h\}} \int_{-\infty}^{\infty} g\left(a_t, e_t\right) \mathrm{d}a. \tag{D.23}$$

The system of equations (D.21)-(D.23) is discretized as follows:

$$0 = -\left[s_{e,i} g_{e,i}\right]' - \phi_{-ee} g_{e,i} - \phi_{e,-e} g_{-e,i} \tag{D.24}$$

$$1 = \sum_{e_t \in \{e_l, e_h\}} \sum_{i=1}^{I} g_{e,i} \Delta a. \tag{D.25}$$

where $g_{e,i} \equiv g\left(a_i, e\right)$. To approximate the derivative $\left[s_{e,i} g_{e,i}\right]'$ we use the upwind differentiation scheme:

$$\left[s_{e,i} g_{e,i}\right]' = \frac{\left(s_{e,i}\right)^+ g_{e,i} - \left(s_{e,i-1}\right)^+ g_{e,i-1}}{\Delta a} + \frac{\left(s_{e,i+1}\right)^- g_{e,i+1} - \left(s_{e,i}\right)^- g_{e,i}}{\Delta a},$$

where $s_{e,i} = r a_i + w e - \left(u'\right)^{-1}\left(V'_{e,i}\right)$ is the optimal savings function obtained from the solution to the HJB equation. Equation (D.24) defines a system of $2 \times I$ linear equations in $g_{e,i}$ with matrix representation:

$$\mathbf{Bg} = \mathbf{0} \tag{D.26}$$

where $\mathbf{g} = \left(g_{e_l,1}, \ldots, g_{e_l,I}, g_{e_h,1}, \ldots, g_{e_h,I}\right)'$. The matrix $\mathbf{B}$ is defined as $\mathbf{B} = \tilde{\mathbf{A}}^\top$, where $\tilde{\mathbf{A}} = -\mathbf{A} + \left(\rho + \frac{1}{\Delta}\right)\mathbf{I}$. The matrix $\tilde{\mathbf{A}}$ captures the evolution of the continuous-time stochastic processes $\{a_t, e_t\}_{t=0}^{\infty}$. To impose the integrability condition in (D.23) we follow Achdou et al. (2020) and fix $g_{e,i} = 0.1$ for an arbitrary i. Then solve the system of equations in (D.26) for some $\tilde{\mathbf{g}}$, and proceed to re-normalize $g_{e,i} = \tilde{g}_{e,i} / \left(\sum_{e,i} \tilde{g}_{e,i} \Delta a\right)$.

E Sensitivity of the wealth distribution

For the model in Section 2, the probability density function of wealth can be obtained from the subdensity functions that solve the Fokker-Plank equations (2.14) and (2.15). Using the identity in (2.12) the (marginal) stationary probability density function of wealth is then given by

$$g\left(a \mid \boldsymbol{\theta}\right) = g\left(a, e_l \mid \boldsymbol{\theta}\right) + g\left(a, e_h \mid \boldsymbol{\theta}\right), \tag{E.27}$$

where $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^{\mathcal{M}}$ denotes the $\mathcal{M} \times 1$ vector of structural parameters in the model, and where $\boldsymbol{\Theta}$ is the parameter space, assumed to be compact. The population values of the structural parameters of the model, $\boldsymbol{\theta}_0$, are those given in Table 1 in the main text.

Since the probability density function is the central component of the maximum likelihood estimator's objective function (see Equations (3.1) and (3.3) in the main text), examining its behavior provides important information on whether it is possible to identify the model parameters using the likelihood of the data. In particular, this Appendix investigate whether it is possible (or not) to distinguish the model's implied density function of wealth approximated using the true parameter values, $g\left(a \mid \boldsymbol{\theta}_0\right)$, from the density function approximated using a range of parameter values that differ from those in the population, $g\left(a \mid \boldsymbol{\theta}\right)$, with $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. In other words, we are interested in studying the sensitivity of the probability density function of wealth when the sampling process is known.

Figure E1 plots the density function of wealth for each of the parameters of the model. In each plot, we perturb within an economically reasonable range the parameter under consideration below and above its population value. The remaining parameters are kept at their true values in the population. The figure reveals that the density function of wealth is sensitive to small changes in the subjective discount factor, ρ, the capital share in output, α, the depreciation rate of capital, δ. On the other hand, changes in the coefficient of relative risk aversion, γ, and the parameters describing the income process, $\phi\left(e\right)$ and e, do not impact change the model's distribution of wealth. As discussed in the main text, we interpret the latter as a source of weak identification problems given that different parameter values deliver the same density function.

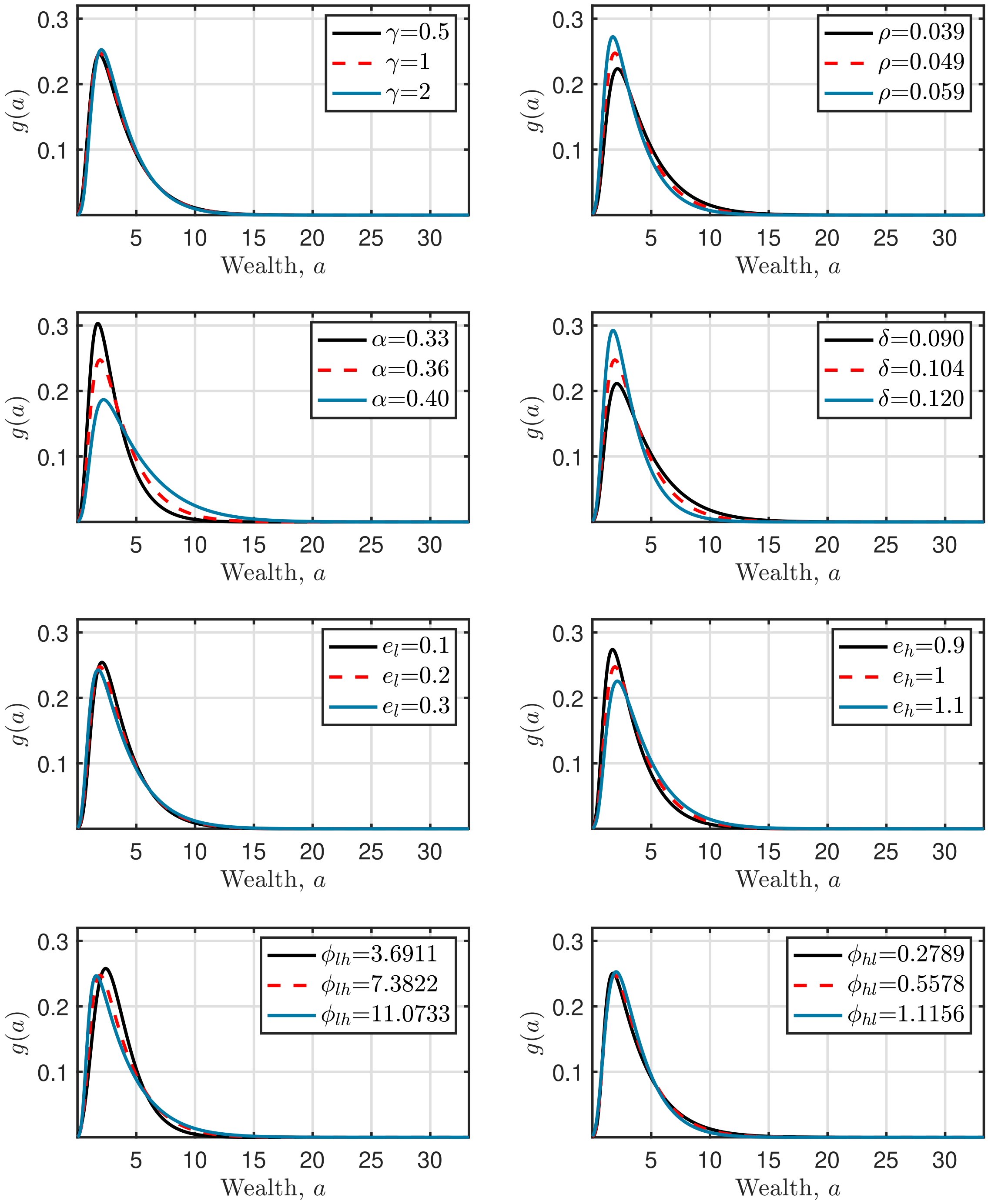

Figure E1. Sensitivity of the wealth distribution. The graph shows the sensitivity of the distribution of wealth, $g\left(a \mid \boldsymbol{\theta}\right)$, for selected parameters $\theta \in \boldsymbol{\theta}$. The dashed line denotes the population density of wealth. The continuous lines correspond to the density of wealth resulting from small perturbations in each parameter while keeping the remaining ones at their true value.

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